INTRODUCTION TO SPECTRAL THEORY OF HANKEL AND TOEPLITZ OPERATORS

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ABSTRACT. These are the notes of the lecture course given at LTCC in 2015. The aim of the course is to consider the following three classes of operators: Toeplitz and Hankel operators on the Hardy space on the unit circle and Toeplitz operators on the Bergman space on the unit disk. For each of these three classes of operators, we consider the following questions: boundedness and estimates or explicit expressions for the norm; compactness; essential spectrum; operators of the finite rank.

1. INTRODUCTION

1.1. $L^p(\mathbb{T})$ **spaces.** We denote by $\mathbb{T}$ the unit circle on the complex plane, parameterised by $e^{i\theta}$, $\theta \in (-\pi, \pi]$ and equipped with the normalised Lebesgue measure $d\theta/2\pi$. Elements f of $L^p(\mathbb{T})$ can be written either as $f(z)$, $|z| = 1$ or as $f(e^{i\theta})$, $|\theta| < \pi$. The norm of f in $L^p(\mathbb{T})$ will be denoted by $\|f\|_p$,

$$\|f\|_p^p = \int_{-\pi}^{\pi} |f(e^{i\theta})|^p \frac{d\theta}{2\pi}.$$

We will be mostly interested in the space $L^2(\mathbb{T})$, which is a Hilbert space. In the case $p = 2$, we will drop the subscript of the norm: $\|f\| = \|f\|_2$. The set $\{z^n\}_{n\in\mathbb{Z}}$ is an orthonormal basis in L^2, so every f can be represented as

$$f(z) = \sum_{n\in\mathbb{Z}} \hat{f}(n) z^n, \quad z \in \mathbb{T},$$

where $\{\hat{f}(n)\}_{n\in\mathbb{Z}}$ is the (two-sided) sequence of the Fourier coefficients of f,

$$\hat{f}(n) = \int_{-\pi}^{\pi} f(e^{i\theta}) e^{-in\theta} \frac{d\theta}{2\pi}.$$

We have the Parseval identity

$$\|\hat{f}\|_{\ell^2}^2 = \sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2 = \int_{-\pi}^{\pi} |f(e^{i\theta})|^2 \frac{d\theta}{2\pi} = \|f\|_{L^2(\mathbb{T})}^2.$$

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1.2. **The harmonic extension.** The *harmonic extension* of $f \in L^1(\mathbb{T})$ is a function of $z \in \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, defined by

$$\widetilde{f}(z) = \sum_{n\geq 0} \hat{f}(n) z^n + \sum_{n\geq 1} \hat{f}(-n)\overline{z}^n = \int_{-\pi}^{\pi} \frac{1-|z|^2}{|1-e^{-i\theta}z|^2} f(e^{i\theta}) \frac{d\theta}{2\pi}, \quad z \in \mathbb{D}.$$

The integral kernel above can be written as the series

$$\frac{1-|z|^2}{|1-e^{-i\theta}z|^2} = \sum_{n\geq 0} e^{-in\theta} z^n + \sum_{n\geq 1} e^{in\theta}\overline{z}^n.$$

In particular, for $0 < r < 1$, we will denote

$$f_r(e^{i\theta}) = \widetilde{f}(re^{i\theta}) = \int_{-\pi}^{\pi} P_r(\theta - t) f(e^{it}) \frac{dt}{2\pi},$$

where $P_r(\theta)$ is the *Poisson kernel*,

$$P_r(\theta) = \frac{1-r^2}{|1-re^{i\theta}|^2} = \sum_{n\geq 0} r^n e^{in\theta} + \sum_{n\geq 1} r^n e^{-in\theta}.$$

The Poisson kernel satisfies the following properties:
(i) $P_r(\theta) > 0$ for all $0 < r < 1$ and all $|\theta| \leq \pi$;
(ii) $\int_{-\pi}^{\pi} P_r(\theta) d\theta / 2\pi = 1$ for all r;
(iii) for any $\delta > 0$, $\int_{|\theta|>\delta} P_r(\theta) d\theta \to 0$ as $r \to 1$.

Proposition 1.1. *The map $f \mapsto f_r$ is a contraction (i.e. it has norm ≤ 1) on $L^p(\mathbb{T})$ for all $1 \leq p \leq \infty$. Further, we have*

$$\|f_r - f\|_p \to 0, \quad r \to 1, \quad \forall f \in L^p(\mathbb{T}), \quad p < \infty. \tag{1.1}$$

For $f \in C(\mathbb{T})$, the convergence $f_r \to f$ is uniform on $\mathbb{T}$.

The proof is outlined in exercises.

There are also results about almost-everywhere pointwise convergence $f_r \to f$, but they are more advanced and we will not need them.

1.3. **The Hardy classes.** For $1 \leq p \leq \infty$, the *Hardy class* $H^p_+ = H^p_+(\mathbb{T})$ is defined as

$$H^p_+(\mathbb{T}) = \{f \in L^p(\mathbb{T}) : \hat{f}(n) = 0, \quad n < 0\}. \tag{1.2}$$

We will only need the cases $p = 1, 2, \infty$. One also defines

$$H^p_-(\mathbb{T}) = \{f \in L^p(\mathbb{T}) : \hat{f}(n) = 0, \quad n \geq 0\}.$$

There is a lack of complete symmetry between H^p_+ and H^p_-, since the constant function belongs to H^p_+ but not to H^p_-.

The functions $f \in H^p_+$ have a natural analytic extension into the unit disk,

$$\widetilde{f}(z) = \sum_{n\geq 0} \hat{f}(n) z^n = \int_{-\pi}^{\pi} \frac{f(e^{i\theta})}{1-ze^{-i\theta}} \frac{d\theta}{2\pi}, \quad z \in \mathbb{D}.$$

This formula can be written as

$$\widetilde{f}(z) = (f, k_z), \quad k_z(e^{i\theta}) = \frac{1}{1 - \overline{z}e^{i\theta}},$$

where $(\cdot, \cdot)$ is the inner product in $L^2(\mathbb{T})$, and k_z is called *the reproducing kernel.*

It is clear that for $f \in H^p_+$, its harmonic extension coincides with the analytic extension. We will use the same notation $\widetilde{f}$ and $f_r(e^{i\theta}) = f(re^{i\theta})$ for the analytic extension. As a corollary of (1.1), we have

$$\|f_r - f\|_p \to 0, \quad r \to 1, \quad \forall f \in H^p_+, \quad p < \infty. \tag{1.3}$$

Consider the case $p = 2$. The Hardy space H^2_+ inherits the Hilbert space structure from L^2. We will use the orthogonal projection

$$P_+ : L^2(\mathbb{T}) \to H^2_+(\mathbb{T}), \quad \sum_{n\in\mathbb{Z}} \hat{f}(n)z^n \mapsto \sum_{n\geq 0} \hat{f}(n)z^n.$$

It is easy to see that for $f \in H^p_+$, the norm $\|f_r\|_p$ is monotone increasing in p (see exercises). Using this and (1.1), we obtain

$$\|f\|_p = \lim_{r\to 1} \|f_r\|_p, \quad p < \infty. \tag{1.4}$$

In fact, this is also true for $p = \infty$. The above relation is often used as an alternative definition of the H^p_+ classes; a function f analytic in $\mathbb{D}$ is said to belong to H^p_+, $p < \infty$, if

$$\sup_{r<1} \int_{-\pi}^{\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} < \infty. \tag{1.5}$$

The space $H^\infty(\mathbb{T})$ can be alternatively defined as the space of all bounded analytic functions on the unit disk. Relation (1.4) shows that any function that belongs to H^p_+ in the sense of (1.2), also satisfies (1.5). The converse implication is not as straightforward, unless $p = 2$ (see exercises).

1.4. **The Bergman space.** Let $L^2(\mathbb{D})$ be the space of all square integrable functions on the unit disk, equipped with the standard norm,

$$\|f\|^2_{L^2} = \int_{\mathbb{D}} |f(z)|^2 dA(z) = \frac{1}{\pi} \int_{-\pi}^{\pi} \int_0^1 |f(re^{i\theta})|^2 r dr d\theta;$$

here $dA(z)$ is the normalised 2-dimensional Lebesgue measure on the unit disk. *The Bergman space* $A^2 = A^2(\mathbb{D})$ is the subspace of $L^2(\mathbb{D})$ which consists of analytic functions. One can analogously define the spaces $A^p(\mathbb{D})$ for all p, but we will not need them. We will use the orthogonal projection

$$\Pi_+ : L^2(\mathbb{D}) \to A^2,$$

sometimes called the Bergman projection. The set $\{(n+1)^{1/2}z^n\}_{n=0}^\infty$ is an orthonormal basis in A^2; we will call it the standard basis. Using the standard basis, it is easy to see that the projection Π_+ can be expressed via the Bergman kernel $k_w(z) = (1 - z\overline{w})^{-2}$:

$$(\Pi_+ f)(w) = \sum_{n \geq 0} (n+1) w^n \int_{\mathbb{D}} f(z) \overline{z}^n dA(z) = \int_{\mathbb{D}} \frac{f(z)}{(1 - \overline{z}w)^2} dA(z).$$

1.5. **The multiplication operators.** Let $a \in L^\infty(\mathbb{T})$. We denote by $M(a)$ the multiplication operator on $L^2(\mathbb{T})$:

$$(M(a)f)(z) = a(z)f(z), \quad z \in \mathbb{T}.$$

We will call a the *symbol* of $M(a)$. It is clear that $M(a)$ is bounded, and $\|M(a)\| \leq \|f\|_\infty$ (we will soon see that in fact $\|M(a)\| = \|f\|_\infty$). It is also obvious that $M(a)^* = M(\overline{a})$, and in particular $M(a)$ is self-adjoint for real-valued symbols.

The Fourier transform maps $M(a)$ into a discrete convolution operator in $\ell^2(\mathbb{Z})$:

$$\begin{aligned}(M(a)f)_n &= \int_{-\pi}^{\pi} a(e^{i\theta}) f(e^{i\theta}) e^{-in\theta} \frac{d\theta}{2\pi} \\ &= \sum_{m \in \mathbb{Z}} \hat{f}(m) \int_{-\pi}^{\pi} a(e^{i\theta}) e^{i(m-n)\theta} \frac{d\theta}{2\pi} = \sum_{m \in \mathbb{Z}} \hat{a}(n-m) \hat{f}(m). \quad (1.6)\end{aligned}$$

Similarly, let $a \in L^\infty(\mathbb{D})$; one can define the multiplication operator $M(a)$ on $L^2(\mathbb{D})$ by

$$(M(a)f)(z) = a(z)f(z), \quad z \in \mathbb{D}.$$

Again, we have the obvious estimate $\|M(a)\| \leq \|f\|_\infty$ (in fact, $\|M(a)\| = \|f\|_\infty$). In contrast to (1.6), in general there is no simple matrix representation for multiplication operators on $L^2(\mathbb{D})$, because there is no simple basis on $L^2(\mathbb{D})$.

1.6. **Toeplitz operators on the Hardy space.** For a symbol $a \in L^\infty(\mathbb{T})$, the *Toeplitz operator* $T(a)$ on $H^2_+(\mathbb{T})$ is defined as

$$T(a)f = P_+ M(a) f = P_+(af), \quad f \in H^2_+(\mathbb{T}).$$

$T(a)$ is sometimes called Hardy-Toeplitz operator. It is clear that $T(a)$ is bounded and

$$\|T(a)\| \leq \|a\|_\infty.$$

The Fourier transform maps $T(a)$ onto the class of *matrix Toeplitz operators* on $\ell^2(\mathbb{Z}_+)$; these are infinite matrices of the type $\{a_{n-m}\}_{n,m \geq 0}$. Indeed, as a consequence of (1.6), we have

$$(T(a)f)_n = \sum_{m \geq 0} \hat{a}(n-m) \hat{f}(m).$$

Clearly, $T(a)^* = T(\overline{a})$ and so, in particular, $T(a)$ is self-adjoint for real-valued symbols a.

1.7. **Toeplitz operators on the Bergman space.** For a symbol $a \in L^\infty(\mathbb{D})$, the *Toeplitz operator* $T(a)$ on $A^2(\mathbb{D})$ is defined as

$$T(a)f = \Pi_+ M(a)f = \Pi_+(af), \quad f \in A^2(\mathbb{D}).$$

$T(a)$ is sometimes called Bergman-Toeplitz operator. It is clear that $T(a)$ is bounded and

$$\|T(a)\| \leq \|a\|_\infty.$$

Bergman-Toeplitz operator can be represented by an infinite matrix in the standard basis $\{(n+1)^{1/2}z^n\}_{n\geq 0}$ of the Bergman space:

$$\begin{aligned} a_{nm} &= (n+1)^{1/2}(m+1)^{1/2}(T(a)z^n, z^m)_{L^2(\mathbb{D})} \\ &= (n+1)^{1/2}(m+1)^{1/2}\frac{1}{\pi}\int_{-\pi}^{\pi}\int_0^1 a(re^{i\theta})r^{n+m+1}e^{i(n-m)\theta}drd\theta. \end{aligned}$$

In contrast to the Hardy-Toeplitz case, the matrix $\{a_{nm}\}_{n,m\geq 0}$ in general does not have any simple structure. Important interesting particular cases when the structure of this matrix simplifies, are

- $a(z) = a(|z|)$ — radial symbols;
- $a(z) = a_1(r)a_2(e^{i\theta})$, $z = re^{i\theta}$ — factorizable symbols.

1.8. **Hankel operators on the Hardy space.** Let J on $L^2(\mathbb{T})$ be the involution:

$$(Jf)(e^{i\theta}) = f(e^{i\theta}).$$

If $f \in H^2_+$ and $\hat{f}(0) = 0$, then $Jf \in H^2_-$. For a symbol $a \in L^\infty(\mathbb{T})$, the *Hankel operator* $H(a)$ on H^2_+ is defined by

$$H(a)f = P_+M(a)Jf = P_+(aJf), \quad f \in H^2_+(\mathbb{T}).$$

Clearly, we have the norm bound

$$\|H(a)\| \leq \|a\|_\infty.$$

The Fourier transform unitarily maps Hankel operators into the class of *matrix Hankel operators* on $\ell^2(\mathbb{Z}_+)$; these are matrices of the type $\{a_{n+m}\}_{n,m\geq 0}$. Indeed,

$$(H(a)z^n, z^m) = (P_+aJz^n, z^m) = (az^{-n}, z^m) = \hat{a}(n+m).$$

It is easy to see that $H(a)^* = H(a_*)$, where $a_*(e^{i\theta}) = \overline{a(e^{-i\theta})}$. In particular, symbols that satisfy the symmetry condition $a = a_*$ generate self-adjoint Hankel operators.

Remark. There are also two types of Hankel operators studied on Bergman space: "big Hankel" and "little Hankel". We will not discuss these in this course.

1.9. **The aim of the course.** For each of the above four classes of operators (multiplication operators, Hardy-Toeplitz, Bergman-Toeplitz, Hankel), we will address the following questions.

- Is the symbol uniquely defined by the operator?
- What are the sufficient (necessary?) conditions in terms of the symbol for the boundedness of the operator? Is there a simple expression for the norm?
- What are the sufficient (necessary?) conditions for the operator to be compact? To be trace class? To be of the finite rank?
- Is there a simple description of the spectrum of the operator?
- For non-compact operators, is there a simple description of the essential spectrum?

The multiplication operators is the simplest class of all four; for multiplication operators, complete answers to all of the above questions are readily available. For the other three classes, some of the questions above turn out to be very non-trivial.

1.10. **Exercises.**

Exercise 1.1. Using the properties (i), (ii) of the Poisson kernel, prove that the map $f \mapsto f_r$ is a contraction in $L^p(\mathbb{T})$ for all $1 \leq p \leq \infty$. *Hint:* for $p = 1$ and $p = \infty$ this is a straightforward calculation. For $1 < p < \infty$, let q be the dual exponent, $1/p + 1/q = 1$; using the Hölder inequality, prove the estimate

$$\left| \int_{-\pi}^{\pi} f_r(e^{i\theta}) g(e^{i\theta}) \frac{d\theta}{2\pi} \right| \leq \|f\|_p \|g\|_q,$$

and use the fact that L^q is the dual space to L^p.

Exercise 1.2. Using the properties (i)–(iii) of the Poisson kernel, prove that for any $f \in C(\mathbb{T})$, we have $\|f_r - f\|_\infty \to 0$ as $r \to 1$. *Hint:* write

$$f_r(e^{i\theta}) - f(e^{i\theta}) = \int_{-\pi}^{\pi} P_r(\theta - t)(f(e^{it}) - f(e^{i\theta})) \frac{dt}{2\pi}.$$

Split the interval of integration into the one where $|\theta - t| < \delta$ and its complement. Use the uniform continuity of f and the property (iii) to estimate each of the two integrals resulting from this split.

Exercise 1.3. Using the above two exercises, prove (1.1). (*Hint: C* is dense in L^p for $p < \infty$.) Show that (1.1) is false for $p = \infty$. (*Hint:* f_r is continuous for all $r < 1$.)

Exercise 1.4. Let $f \in L^p(\mathbb{T})$. Prove that $\|f_r\|_p$ is monotone increasing in r. *Hint:* $(f_{r_1})_{r_2} = f_{r_1 r_2}$.

Exercise 1.5. Let f be a function analytic in the unit disk, satisfying (1.5) with $p = 2$. Prove that $f \in H^2_+$ in the sense of (1.2). *Hint:* write the Taylor series of f,

$$f(z) = \sum_{n=0}^{\infty} c_n z^n,$$

and using (1.5), prove that the sequence $\{c_n\}_{n=0}^{\infty}$ is in ℓ^2.

2. Multiplication operators

Here we consider the multiplication operators $M(a)$ on $L^2(\mathbb{T})$ and on $L^2(\mathbb{D})$.

2.1. **Uniqueness of the symbol.** Let us prove that the symbol a is uniquely determined by the operator $M(a)$, i.e. $M(a_1) = M(a_2)$ implies $a_1 = a_2$ almost everywhere. It suffices to prove that $M(a) = 0$ implies $a = 0$ almost everywhere. Suppose to the contrary that $a \neq 0$ on some set $E \subset \mathbb{T}$ (resp. $E \subset \mathbb{D}$) of a positive Lebesgue measure. Let χ_E be the characteristic function of E. Then $\|M(a)\chi_E\| \neq 0$, contrary to the assumption $M(a) = 0$. This contradiction proves the claim.

2.2. **The norm.** The norm bound

$$\|M(a)\| \leq \|a\|_\infty$$

is obvious. Let us prove the opposite inequality:

$$\|a\|_\infty \leq \|M(a)\|.$$

Let's assume $\|a\|_\infty > 0$ (otherwise there is nothing to prove). For any sufficiently small $\varepsilon > 0$, there exists a set $E_\varepsilon \subset \mathbb{T}$ (resp. $E_\varepsilon \subset \mathbb{D}$) of a positive Lebesgue measure such that $|a(z)| > \|a\|_\infty - \varepsilon$ for $z \in E_\varepsilon$. Now take $f = \chi_{E_\varepsilon}$; then

$$|(M(a)f)(z)| \geq (\|a\|_\infty - \varepsilon)|f(z)|,$$

and so $\|M(a)f\| \geq (\|a\|_\infty - \varepsilon)\|f\|$. It follows that $\|M(a)\| \geq \|a\|_\infty - \varepsilon$; since ε can be taken arbitrarily small, we obtain $\|a\|_\infty \leq \|M(a)\|$.

By a similar argument, it is easy to see that the inclusion $a \in L^\infty$ is not only sufficient, but also necessary condition for the boundedness of $M(a)$.

2.3. **Non-compactness.** Let us prove that the operator $M(a)$ is compact only if $a = 0$ almost everywhere.

The proof is very easy in the case of the operator of multiplication in $L^2(\mathbb{T})$. If $M(a)$ is compact, then it maps weakly convergent sequences to strongly convergent ones. Consider the sequence $f_n(z) = z^n$, $n \in \mathbb{N}$ in $L^2(\mathbb{T})$; this sequence converges weakly to zero, and $\|M(a)f_n\|_2 = \|a\|_2 \neq 0$ unless $a = 0$.

Consider the case of the operator in $L^2(\mathbb{D})$. Suppose $a \neq 0$; then there exists $\varepsilon > 0$ and a set $E \subset \mathbb{D}$ of a positive Lebesgue measure such that $|a(z)| > \varepsilon$ for all $z \in E$. Let us choose a disjoint sequence of sets $E_n \subset E$ of a positive Lebesgue measure. Consider $f_n = \chi_{E_n}/\|\chi_{E_n}\|$. Then f_n converges weakly to zero (exercise). But we have

$$\|M(a)f_n\|^2 = \frac{1}{|E_n|}\int_{E_n} |a(x)|^2 \geq \varepsilon$$

(here $|E_n|$ is the Lebesgue measure of E_n). So $M(a)f_n$ does not converge strongly to zero, which contradicts the compactness of $M(a)$.

2.4. **The spectrum of $M(a)$.** Let $\mathcal{H}$ be a separable Hilbert space; we denote by $\mathcal{B}(\mathcal{H})$ the set of all bounded linear operators on $\mathcal{H}$.

For $M \in \mathcal{B}(\mathcal{H})$, the *spectrum of* M, denoted by $\sigma(M)$, is the set of all $\lambda \in \mathbb{C}$ such that the operator $M - \lambda = M - \lambda I$ is not invertible. Recall that $M - \lambda$ is invertible if and only if

$$\operatorname{Ker}(M - \lambda) = \{0\} \quad \text{and} \quad \operatorname{Ran}(M - \lambda) = \mathcal{H}.$$

The 'only if' part is trivial, and the 'if' part is the consequence of the deep Banach inverse map theorem.

If $\lambda \in \sigma(M)$, then at least one of the following is true:

- $\operatorname{Ker}(M - \lambda) \neq \{0\}$, i.e. λ is an eigenvalue;
- $(\operatorname{Ran}(M - \lambda))^\perp \neq \{0\}$, i.e. $\operatorname{Ran}(M - \lambda)$ is not dense in $\mathcal{H}$;
- $\operatorname{Ran}(M - \lambda)$ is not closed.

In the finite dimensional case $\dim \mathcal{H} < \infty$, we have $\lambda \in \sigma(M)$ if and only if $\operatorname{Ker}(M - \lambda) \neq \{0\}$; in the infinite dimensional case, the situation is more complex.

Lecture 2:

Let us determine the spectrum of the multiplication operators. For a symbol a, denote by $\mathcal{R}(a)$ the *essential range* of a, i.e. the set of all $\lambda \in \mathbb{C}$ such that the set

$$E_\varepsilon = \{z : |a(z) - \lambda| < \varepsilon\}$$

has a positive Lebesgue measure. For multiplication operators both on $L^2(\mathbb{T})$ and on $L^2(\mathbb{D})$, we have

Theorem 2.1. *For any $a \in L^\infty$, we have*

$$\sigma(M(a)) = \mathcal{R}(a).$$

Proof. Suppose $\lambda \notin \mathcal{R}(a)$; let us prove that $\lambda \notin \sigma(M(a))$. By the definition of the essential range, for some $\varepsilon > 0$ the set E_ε has measure zero. It follows that the function $b(z) = 1/(a(z) - \lambda)$ is in L^∞, and $\|b\|_\infty \leq 1/\varepsilon$. Now it is easy to check that $M(b)$ is the inverse of $M(a) - \lambda = M(a - \lambda)$.

Conversely, suppose $\lambda \notin \sigma(M(a))$; let us prove that $\lambda \notin \mathcal{R}(a)$. Subtracting a constant, we can always reduce the problem to the case $\lambda = 0$. The statement $0 \notin \sigma(M(a))$ means that $M(a)$ is invertible, and so for any $f \in L^2(\mathbb{T})$,

$$\|f\| = \|M(a)^{-1}M(a)f\| \leq \|M(a)^{-1}\|\|af\|.$$

From here it is easy to conclude (see Exercise 2.2) that $|a| \geq 1/\|M(a)^{-1}\|$ a.e. on $\mathbb{T}$, and so taking $\varepsilon < 1/\|M(a)^{-1}\|$ in the definition of the essential range, we conclude that $0 \notin \mathcal{R}(a)$. □

2.5. **Essential spectrum: background.**

Definition. An operator $M \in \mathcal{B}(\mathcal{H})$ is called *Fredholm*, if $\operatorname{Ran} M$ is closed and $\dim \operatorname{Ker} M < \infty$, $\dim \operatorname{Ker} M^* < \infty$.

Remark. (1) If M is invertible, then $\operatorname{Ker} M = \{0\}$ and $\operatorname{Ran} M = \mathcal{H}$. Due to the formula

$$\overline{\operatorname{Ran} M} \oplus \operatorname{Ker} M^* = \mathcal{H}, \tag{2.1}$$

we also have $\operatorname{Ker} M^* = \{0\}$. Thus, every invertible operator is Fredholm.

(2) If $\dim \mathcal{H} < \infty$, then any bounded operator M on $\mathcal{H}$ is Fredholm.

Proposition 2.2. *An operator $M \in \mathcal{B}(\mathcal{H})$ is Fredholm iff there exists $R \in \mathcal{B}(\mathcal{H})$ such that $RM - I$, $MR - I$ are compact.*

See e.g. Section 4.3 of the book *Linear Operators and Their Spectra* by E.B.Davies.

Corollary 2.3. (i) *A compact operator is never Fredholm unless $\dim \mathcal{H} < \infty$.*
(ii) *M is Fredholm iff M^* is Fredholm.*
(iii) *If M is Fredholm, then $M + K$ is Fredholm for any compact operator K.*
(iv) *If M is invertible, and K is compact, then $M+K$ is Fredholm. In particular, $I + K$ is Fredholm.*

Proof. (i) Suppose M is a compact Fredholm operator. Then $RM - I = K$, where K is compact. It follows that I is compact; this is possible only if $\dim \mathcal{H} < \infty$.
(ii) If $RM - I$, $MR - I$ are compact, then $R^*M^* - I$, $M^*R^* - I$ are compact.
(iii) If $RM - I$ is compact, then $R(M+K) - I = RM + RK - I$ is also compact.
(iv) Follows directly from (iii). □

Part (iii) of this Corollary shows that Fredholmness is a stable property with respect to compact perturbations. In fact, it is also a stable property with respect to perturbations of a small norm:

Proposition 2.4. *Let M be Fredholm. Then there exists $\varepsilon > 0$ such that for any $T \in \mathcal{B}(\mathcal{H})$ with $\|T\| < \varepsilon$, the operator $M + T$ is Fredholm.*

See Davies' book for the proof.

Definition. Let $M \in \mathcal{B}(\mathcal{H})$. The *essential spectrum* of M is defined as

$$\sigma_{\text{ess}}(M) = \{\lambda \in \mathbb{C} : M - \lambda \text{ is not Fredholm.}\}$$

Remark. (1) There are several distinct *non-equivalent* definitions of the essential spectrum in the literature. For normal operators, they all coincide.

(2) Clearly, $\sigma_{\text{ess}}(M) \subset \sigma(M)$.

(3) $\lambda \in \sigma_{\text{ess}}(M)$ iff at least one of the following holds:
- $\dim \operatorname{Ker}(M - \lambda) = \infty$;
- $\dim(\operatorname{Ran}(M - \lambda))^{\perp} = \infty$;
- $\operatorname{Ran}(M - \lambda)$ is not closed.

Compare this with the situation when $\lambda \in \sigma(M)$.

(4) If M is an operator in the finite dimensional space, then $\sigma_{\text{ess}}(M) = \varnothing$.

(5) For compact operators M (in infinite dimensional Hilbert space) we have $\sigma_{\text{ess}}(M) = \{0\}$. Indeed, $0 \in \sigma_{\text{ess}}(M)$ follows from Corollary 2.3(i). If $\lambda \neq 0$, then $M - \lambda I = (-\lambda)(I - M/\lambda)$ is Fredholm by Corollary 2.3(iv).

(6) By Proposition 2.4, $\sigma_{\text{ess}}(M)$ is a closed set.

(7) By Corollary 2.3(iii), the essential spectrum is stable under compact perturbations: $\sigma_{\text{ess}}(M) = \sigma_{\text{ess}}(M + K)$ for compact K.

2.6. **Essential spectrum of multiplication operators.**

Theorem 2.5. *For any symbol $a \in L^\infty$, we have*

$$\sigma_{ess}(M(a)) = \sigma(M(a)) = \mathcal{R}(a).$$

Let us first prove a simple

Lemma 2.6. *For any $a \in L^\infty$, we have either* $\operatorname{Ker} M(a) = \{0\}$ *or* $\dim \operatorname{Ker} M(a) = \infty$.

Proof. $M(a)f = 0$ means that $\operatorname{supp} f \subset E$, where $E = \{z : a(z) = 0\}$. If $|E| = 0$, then $f = 0$, and if $|E| > 0$, then $\dim \operatorname{Ker} M(a) = \infty$. □

Proof of Theorem 2.5. By Theorem 2.1, it suffices to prove the inclusion $\sigma(M(a)) \subset \sigma_{\text{ess}}(M(a))$. Suppose $\lambda \notin \sigma_{\text{ess}}(M(a))$; let us prove that $\lambda \notin \sigma(M(a))$. It suffices to consider the case $\lambda = 0$. If $0 \notin \sigma_{\text{ess}}(M(a))$, then $M(a)$ is Fredholm, so $\operatorname{Ran} M(a)$ is closed and $\operatorname{Ker} M(a)$ and $\operatorname{Ker} M(\overline{a})$ are finite dimensional. By the previous Lemma, $\operatorname{Ker} M(a) = \{0\}$ and $\operatorname{Ker} M(\overline{a}) = \{0\}$. Thus, we also get $\operatorname{Ran} M(a) = \mathcal{H}$. By the Banach's inverse mapping theorem, it follows that $M(a)$ is invertible, so $0 \notin \sigma(M(a))$. □

2.7. **Exercises.**

Exercise 2.1. Prove that the sequence f_n from the proof of non-compactness converges weakly to zero. *Hint:* use Cauchy-Schwarz and the fact that for $g \in L^1$, one has $\sum_n \int_{E_n} g(x)dx < \infty$.

Exercise 2.2. Let $a \in L^\infty(\mathbb{T})$; suppose that there exists a constant $C > 0$ such that for any trigonometric polynomial f on $\mathbb{T}$ (i.e. for any function of the type $f(z) = \sum_{|n| \leq N} f_n z^n$, $z \in \mathbb{T}$) we have

$$\|af\|_2 \geq C\|f\|_2.$$

Prove that $\operatorname{ess\,inf}_{\mathbb{T}}|a| \geq C$. *Hint:* Trigonometric polynomials are dense in L^2. Any characteristic function of a measureable set is in L^2. Now argue by contradiction.

The purpose of the following two exercises is to give an alternative proof of the inclusion $\mathcal{R}(a) \subset \sigma_{\text{ess}}(M(a))$.

Exercise 2.3. Let $M \in \mathcal{B}(\mathcal{H})$. Suppose that there exists a sequence $\{f_n\}_{n=1}^{\infty}$ in $\mathcal{H}$ such that $\inf_n \|f_n\| > 0$, $f_n \to 0$ weakly and $\|(M - \lambda)f_n\| \to 0$ as $n \to \infty$ (such sequence is called a *Weyl sequence* for λ). Prove that $\lambda \in \sigma_{\text{ess}}(M)$.

Exercise 2.4. Let $a \in L^\infty$, suppose $\lambda \in \mathcal{R}(a)$. Construct a Weyl sequence for the operator $M(a)$ and the point λ and thus prove that $\lambda \in \sigma_{\text{ess}}(M(a))$. *Hint:* your Weyl sequence should consists of the characteristic functions of an appropriate family of sets.

3. Toeplitz operators on Hardy space

Notation: Let $a \in L^1(\mathbb{T})$. We will denote by $\widetilde{a}$ the harmonic extension of a into $\mathbb{D}$. In particular, if $a \in H^2_+$, then $\widetilde{a}$ is the analytic extension of a.

3.1. Uniqueness of the symbol.

Theorem 3.1. *Let $T(a_1) = T(a_2)$ for $a_1, a_2 \in L^\infty$. Then $a_1 = a_2$ a.e. on $\mathbb{T}$.*

Proof. It suffices to prove that $T(a) = 0$ implies $a = 0$. As we have already seen,

$$(T(a)z^n, z^m) = \hat{a}(n - m)$$

for all $n, m \geq 0$. It follows that all Fourier coefficients of a vanish, so $a = 0$. □

3.2. The norm.

Theorem 3.2. *For any $a \in L^\infty(\mathbb{T})$, one has*

$$\|T(a)\| = \|a\|_\infty.$$

As we have the obvious bound $\|T(a)\| \leq \|a\|_\infty$, it suffices to prove that

$$\|a\|_\infty \leq \|T(a)\|.$$

We will give two proofs of the last bound: by using some simple functional analysis and by using some analytic function theory. For the first proof, we need

Lemma 3.3. *Let A_n be a sequence of bounded operators on a Hilbert space which converges weakly to an operator A. Then*

$$\|A\| \leq \liminf_{n\to\infty} \|A_n\|.$$

The proof is an exercise.

First proof of Theorem 3.2. As we already know that $\|M(a)\| = \|a\|_\infty$, it suffices to check that

$$\|M(a)\| \leq \|T(a)\|.$$

First observe that

$$\|T(a)\| = \|P_+ M(a) P_+\|,$$

where the operator on the r.h.s. acts on L^2. Next, the operator $M(z)$ of multiplication by z on L^2 is unitary, and therefore

$$\|P_+M(a)P_+\| = \|M(\overline{z})^n P_+M(a)P_+M(z)^n\|$$

for all $n \geq 0$. Further, for any $k, \ell \in \mathbb{Z}$ we easily check that

$$(M(a)P_+M(z)^n z^k, P_+M(z)^n z^\ell) \to (M(a)z^k, z^\ell)$$

as $n \to \infty$. It follows that we have the weak convergence

$$M(\overline{z})^n P_+M(a)P_+M(z)^n \to M(a)$$

as $n \to \infty$. Applying Lemma 3.3, we get

$$\|M(a)\| \leq \liminf_{n\to\infty} \|M(\overline{z})^n P_+M(a)P_+M(z)^n\| = \|P_+M(a)P_+\| = \|T(a)\|,$$

as required. □

For the second proof, we need the following statement:

Lemma 3.4. *Let $a \in L^1(\mathbb{T})$, and suppose that the harmonic extension of a obeys $|\widetilde{a}(w)| \leq C$, $w \in \mathbb{D}$. Then $a \in L^\infty(\mathbb{T})$ and $\|a\|_\infty \leq C$.*

The proof of this lemma is outlined in the exercises.

Second proof of Theorem 3.2. Let k_w be the reproducing kernel for Hardy space,

$$k_w(e^{i\theta}) = \frac{1}{1-\overline{w}e^{i\theta}} = \sum_{n=0}^{\infty} \overline{w}e^{in\theta},$$

then

$$|(T(a)k_w, k_w)| \leq \|T(a)\|\|k_w\|^2.$$

It is easy to see that

$$\|k_w\|^2 = \sum_{n=0}^{\infty} |w|^{2n} = 1 - |w|^2,$$

and that

$$|k_w(e^{i\theta})|^2/\|k_w\|^2 = \frac{1-|w|^2}{|1-\overline{w}e^{i\theta}|^2} = P_r(\theta - t), \quad w = re^{it},$$

where P_r is the Poisson kernel. Thus, the above estimate rewrites as

$$|\widetilde{a}(w)| \leq \|T(a)\|, \quad w \in \mathbb{D}.$$

By Lemma 3.4, we obtain that $\|a\|_\infty \leq \|T(a)\|$, as required. □

3.3. **Non-compactness.**

Theorem 3.5. *$T(a)$ is compact only if $a = 0$.*

Proof. Assume that $T(a)$ is compact. Then it maps weakly convergent sequences to strongly convergent ones. In particular, $\|T(a)z^k\| \to 0$ as $k \to \infty$. On the other hand, for all $n \in \mathbb{Z}$ and all $k \geq 0$ such that $k + n \geq 0$, we have

$$\hat{a}(n) = (T(a)z^k, z^{n+k}).$$

Sending $k \to \infty$, we obtain $\hat{a}(n) = 0$ for all n, so $a = 0$. □

3.4. **The spectrum.** In general, the description of the spectrum of a Toeplitz operator is a difficult problem. We will consider two classes of symbols for which one can make some progress: $a \in H^\infty(\mathbb{T})$ and $a \in C(\mathbb{T})$. We will prove

Theorem 3.6. *Let $a \in H^\infty(\mathbb{T})$; then*

$$\sigma(T(a)) = \operatorname{clos}(\widetilde{a}(\mathbb{D})).$$

Let us start with a technical

Lemma 3.7. (i) *Let $a \in L^2(\mathbb{T})$. Suppose that for some $C > 0$ and for all $f \in H^\infty_+$, we have*

$$\|af\|_2 \leq C\|f\|_2.$$

Then $a \in L^\infty$ and $\|a\|_\infty \leq C$.

(ii) *Let $a \in L^\infty(\mathbb{T})$. Suppose that for some $C > 0$ and for all $f \in H^\infty_+$, we have*

$$\|af\|_2 \geq C\|f\|_2.$$

Then $\operatorname{ess\,inf}_{\mathbb{T}}|a| \geq C$.

Proof. (i) This is essentially the same argument as in the second proof of Theorem 3.2. We have

$$|(ak_w, k_w)| \leq \|ak_w\|_2\|k_w\|_2 \leq C\|k_w\|_2^2,$$

which can be rewritten as the estimate $|\widetilde{a}(w)| \leq C$, $w \in \mathbb{D}$, for the harmonic extension of a. Then the claim follows by Lemma 3.4.

(ii) From the hypothesis we get

$$\|af\overline{z}^n\|_2 \geq \|f\overline{z}^n\|_2, \quad n \geq 0.$$

Thus, we obtain the estimate

$$\|ag\|_2 \geq C\|g\|_2$$

for any trigonometric polynomial g. Now the result follows by Exercise 2.2. □

Proof of Theorem 3.6. (1) Let $\lambda \notin \operatorname{clos}(\widetilde{a}(\mathbb{D}))$; let us prove that $\lambda \notin \sigma(T(a))$. The general case can always be reduced to $\lambda = 0$. If $0 \notin \operatorname{clos}(\widetilde{a}(\mathbb{D}))$, then the function $b(z) = 1/a(z)$ is bounded on $\mathbb{D}$, and so $b \in H^\infty$. Thus, the operator $T(b)$ is bounded. By a direct calculation, one checks that

$$T(a)T(b) = T(b)T(a) = I,$$

and so $T(b)$ is the inverse of $T(a)$. So $0 \notin \sigma(T(a))$, as required.

(2) Let $\lambda \notin \sigma(T(a))$; let us prove that $\lambda \notin \operatorname{clos}(\widetilde{a}(\mathbb{D}))$. Again, it suffices to consider the case $\lambda = 0$. If $T(a)$ is invertible, then $\operatorname{Ran} T(a) = H^2_+$ and in particular this range includes the function identically equal to 1. Thus, there exists $b \in H^2_+$ such that $T(a)b = 1$, i.e. $ab = 1$. It remains to prove that b is bounded. For any $f \in H^\infty_+$, we have $bf \in H^2_+$ and

$$\|bf\|_2 = \|T(a)^{-1}T(a)bf\|_2 \leq \|T(a)^{-1}\|\|T(a)bf\|_2 = \|T(a)^{-1}\|\|abf\|_2 = \|T(a)^{-1}\|\|f\|_2.$$

By Lemma 3.7(i), we get that $b \in H^\infty_+$. □

3.5. **The essential spectrum.**

Theorem 3.8. *Let $a \in C(\mathbb{T})$; then*

$$\sigma_{ess}(T(a)) = a(\mathbb{T}).$$

We will need an important

Lemma 3.9. *Let $a \in L^\infty$ be a non-zero function. Then either* $\operatorname{Ker} T(a) = \{0\}$ *or* $\operatorname{Ker} T(a)^* = \{0\}$ *(or both).*

Proof. Suppose $f \in \operatorname{Ker} T(a)$, $g \in \operatorname{Ker} T(\overline{a})$. Then $P_+(af) = 0$, i.e. $af \in H^2_-$. Similarly, $\overline{a}g \in H^2_-$. It's easy to see that $g \in H^2_+$, $af \in H^2_-$ implies $af\overline{g} \in H^1_-$. Similarly, we obtain $\overline{a}g\overline{f} \in H^1_-$. So for the function $h = af\overline{g}$ we have $h \in H^1_-$, $\overline{h} \in H^1_-$, so $h \equiv 0$. Since $a \not\equiv 0$, it follows that the product $f\overline{g}$ vanishes on a set of positive measure.

Now we use a uniqueness theorem for Hardy classes: if $\omega \in H^1_+$ and $\omega = 0$ on a set of positive measure on $\mathbb{T}$, then $\omega \equiv 0$. It follows that $fg \equiv 0$, and so either $f \equiv 0$ or $g \equiv 0$ (or both). □

Lecture 3:

Lemma 3.10. *For any $a \in L^\infty$, we have the inclusion*

$$\mathcal{R}(a) \subset \sigma_{ess}(T(a)).$$

Proof. Suppose $\lambda \notin \sigma_{\text{ess}}(T(a))$; let us prove that $\lambda \notin \mathcal{R}(a)$. It suffices to consider the case $\lambda = 0$. The statement $0 \notin \sigma_{\text{ess}}(T(a))$ means that $T(a)$ is Fredholm, so $\operatorname{Ran} T(a)$ is closed. By the previous lemma, we have either $\operatorname{Ker} T(a) = \{0\}$ or $\operatorname{Ker} T(a)^* = \{0\}$.

Suppose first that $\operatorname{Ker} T(a) = \{0\}$. Then $T(a)$ is a bijection between H^2_+ and the closed subspace $\operatorname{Ran} T(a)$ in H^2_+. By Banach's inverse mapping theorem, $T(a)$ has a bounded inverse R on $\operatorname{Ran} T(a)$, so for any $f \in H^2_+$ we have

$$\|f\|_2 = \|RT(a)f\|_2 \leq \|R\|\|T(a)f\|_2 \leq \|R\|\|af\|_2.$$

By Lemma 3.7(ii), we obtain that $\operatorname{ess\,inf}_{\mathbb{T}}|a| > 0$, and so $0 \notin \mathcal{R}(a)$.

If we have $\operatorname{Ker} T(a)^* = \operatorname{Ker} T(\overline{a}) = \{0\}$, then the above argument can be repeated with $\overline{a}$ in place of a. □

We denote by $P_- = I - P_+$ the orthogonal projection onto $H^2_-(\mathbb{T})$ in $L^2(\mathbb{T})$.

Lemma 3.11. *For any $a \in C(\mathbb{T})$, the operator $P_+M(a)P_-$ is compact.*

Proof. First consider the case $a(z) = z^n$. For $n < 0$, we clearly have $P_+M(z^n)P_- = 0$. For $n \geq 0$, we have

$$P_+M(z^n)P_-z^k = 0, \quad \text{if } k < -n,$$

and so $P_+M(z^n)P_-$ is a finite rank operator. It follows that if a is a polynomial, then $P_+M(a)P_-$ is a finite rank operator, so it is compact.

Next, we have the norm estimate

$$\|P_+M(a)P_-\| \leq \|M(a)\| = \|a\|_\infty.$$

Now take a sequence of polynomials a_N such that $\|a - a_N\|_\infty \to 0$; then we obtain

$$\|P_+M(a)P_- - P_+M(a_N)P_-\| \leq \|a - a_N\|_\infty \to 0,$$

and so $P_+M(a)P_-$ has been approximated in the operator norm by compact operators. It follows that $P_+M(a)P_-$ is compact. □

Proof of Theorem 3.8. It is easy to see that for a continuous function a, we have $\mathcal{R}(a) = a(\mathbb{T})$ (see Exercise 3.3). By Lemma 3.10, we obtain the inclusion $a(\mathbb{T}) \subset \sigma_{\mathrm{ess}}(T(a))$. So it remains to prove that

$$\sigma_{\mathrm{ess}}(T(a)) \subset a(\mathbb{T}).$$

Let $\lambda \notin a(\mathbb{T})$; let us prove that $\lambda \notin \sigma_{\mathrm{ess}}(T(a))$. It suffices to consider $\lambda = 0$. Our assumption means that $b = 1/a$ is continuous on $\mathbb{T}$. Let us prove that $T(b)$ is the inverse of $T(a)$ modulo compact operators. For $f \in H^2_+$ we have

$$T(a)T(b)f = P_+aP_+bf = P_+a(I - P_-)bf = P_+abf - P_+aP_-bf = P_+f - P_+aP_-bf.$$

This can be written as

$$T(a)T(b) = I - P_+M(a)P_-M(b),$$

and the operator $P_+M(a)P_-M(b)$ in the r.h.s. is compact by Lemma 3.11. Thus, $T(a)T(b) - I$ is compact, and similarly one proves that $T(b)T(a) - I$ is compact, which means that $T(a)$ is Fredholm, and so $0 \notin \sigma_{\mathrm{ess}}(T(a))$. □

3.6. **The index of a Fredholm operator: background.**

Definition. Let $M \in \mathcal{B}(\mathcal{H})$ be a Fredholm operator. The *index of* M is defined as

$$\operatorname{index}(M) = \dim \operatorname{Ker} M - \dim \operatorname{Ker} M^*.$$

Remark. (1) If M is invertible, then $\operatorname{index} M = \{0\}$. Indeed, we have $\operatorname{Ker} M = \{0\}$ by definition of invertibility, and $\operatorname{Ker} M^* = \{0\}$ follows from formula (2.1).

(2) Any operator M on a finite-dimensional space has index zero. Indeed, from (2.1) we have

$$\dim \operatorname{Ker} M^* + \dim \operatorname{Ran} M = \dim \mathcal{H},$$

and in the finite dimensional case we also have

$$\dim \operatorname{Ker} M + \dim \operatorname{Ran} M = \dim \mathcal{H}.$$

Subtracting, we get $\operatorname{index} M = 0$.

(3) For any compact operator K, we have $\operatorname{index}(I + K) = 0$. This is a non-trivial statement, which we don't prove here.

Example 3.12. Let S be the shift operator on $\ell^2 = \ell^2(\mathbb{Z}_+)$:

$$S : (x_0, x_1, x_2, \dots) \mapsto (0, x_0, x_1, x_2, \dots).$$

The adjoint S^* is the backwards shift:

$$S^* : (x_0, x_1, x_2, \dots) \mapsto (x_1, x_2, x_3, \dots).$$

It is easy to see that $\operatorname{Ker} S = \{0\}$ and $\dim \operatorname{Ker} S^* = 1$. Thus, $\operatorname{index} S = -1$, $\operatorname{index} S^* = 1$. Likewise, $\operatorname{Ker} S^n = \{0\}$ and $\dim \operatorname{Ker}(S^*)^n = n$, and so $\operatorname{index} S^n = -n$, $\operatorname{index}(S^*)^n = n$.

The important property of index is its stability:

Proposition 3.13. *Let M be a Fredholm operator. Then there exists $\varepsilon > 0$ such that for all $T \in \mathcal{B}(\mathcal{H})$ with $\|T\| < \varepsilon$ and for all compact K, the operator $M+T+K$ is Fredholm and*

$$\operatorname{index}(M + T + K) = \operatorname{index}(M).$$

See e.g. Davies' book *Linear operators and their spectra* for the proof.

This statement shows that for any operator M, the function $\operatorname{index}(M - \lambda)$, defined on the complement $(\sigma_{\text{ess}}(M))^c$, is constant on all connected components of this complement.

3.7. **The index of Toeplitz operators with continuous symbols.** We start with an example:

Lemma 3.14. *For $n \in \mathbb{Z}$, we have* $\operatorname{index} T(z^n) = -n$.

Proof. For $n = 0$, $T(z^0) = I$ and so clearly the index is zero. Let $n > 0$; then $T(z^n)$ is unitarily equivalent to S^n in ℓ^2, so $\operatorname{index} T(z^n) = \operatorname{index} S^n = -n$.

Next, for $n > 0$ the operator $T(z^{-n})$ is unitarily equivalent to $(S^*)^n$, and so the index equals n. □

Definition. Let $a \in C(\mathbb{T})$, $a \neq 0$ on $\mathbb{T}$. The *winding number* $\operatorname{wind} a$ of a with respect to the origin is defined as follows. Write

$$a(e^{it}) = |a(e^{it})|e^{i\varphi(t)}, \quad t \in [0, 2\pi],$$

where $\varphi(t)$ is a continuous function of $t \in [0, 2\pi]$. Then

$$\operatorname{wind} a = \frac{1}{2\pi}(\varphi(2\pi) - \varphi(0)).$$

Theorem 3.15. *Let $a \in C(\mathbb{T})$; then for any $\lambda \notin a(\mathbb{T})$,*

$$\operatorname{index}(T(a) - \lambda) = -\operatorname{wind}(a - \lambda).$$

Proof. Set $\lambda = 0$ for simplicity. Let $n = \operatorname{wind} a$; since a does not vanish on $\mathbb{T}$, the curve $\{a(e^{i\theta})\}_{\theta=0}^{2\pi}$ is homotopic to the curve $\{e^{in\theta}\}_{\theta=0}^{2\pi}$. The homotopy can be effected through a family $\{a_\tau\}_{\tau=0}^{1}$ of continuous curves which do not cross the origin. Thus, the family of Toeplitz operators $\{T(a_\tau)\}_{\tau=0}^{1}$ is continuous (in the operator norm) and each operator in the family is Fredholm. Thus, $\operatorname{index} T(a_\tau)$ depends continuously on $\tau \in [0, 1]$; since it is an integer-valued function, the index is constant along this family, hence

$$\operatorname{index} T(a) = \operatorname{index} T(z^n) = -n,$$

as required. □

3.8. Exercises.

Exercise 3.1. Arguing by contradiction, prove Lemma 3.3.

Exercise 3.2. Prove Lemma 3.4. Argue as follows. By Exercise 1.3, we have the convergence $\|a - a_r\|_1 \to 0$. Now suppose the claim is false; then there exists a set $E \subset \mathbb{T}$ of a positive measure and a $\delta > 0$ such that $|a| > C + \delta$ on E. Then $|a - a_r| > \delta$ on E. Use this to get a contradiction.

Exercise 3.3. (i) Prove that for $a \in C(\mathbb{T})$, one has $\mathcal{R}(a) = a(\mathbb{T})$. (ii) Prove that for $H^\infty(\mathbb{T})$, one has $\operatorname{clos} \widetilde{a}(\mathbb{D}) = \mathcal{R}(\widetilde{a})$.

Exercise 3.4. Let $a \in C(\mathbb{T})$. Prove that $a(\mathbb{T}) \subset \sigma_{\mathrm{ess}}(T(a))$ by constructing a Weyl sequence for any $\lambda \in a(\mathbb{T})$. Use the normalised function k_w, where $w = r_n e^{i\theta}$, $r_n \to 1$, converges to a point $e^{i\theta}$ with $a(e^{i\theta}) = \lambda$.

4. Toeplitz operators on Bergman space

4.1. Measures as symbols.

For $a \in L^\infty(\mathbb{D})$, we have the trivial norm estimate

$$\|T(a)\| \leq \|a\|_\infty.$$

In contrast to the Hardy-Toeplitz operators, we will soon see that this estimate is very far from being optimal. Indeed, we will see that there are bounded Toeplitz operators on A^2 with symbols that are not only unbounded, but that are measures rather than functions. Let us introduce some notation required for this case.

Let μ be a finite complex valued measure on $\mathbb{D}$, i.e. $\mu = \mu_1 - \mu_2 + i\mu_3 - i\mu_4$, where μ_j are finite non-negative measures. Consider the sesquilinear (i.e. linear in the first variable and conjugate-linear in the second variable) form

$$t(f,g) = \int_{\mathbb{D}} f(z)\overline{g(z)}d\mu(z)$$

on all analytic polynomials f, g. If t satisfies the bound

$$|t(f,g)| \leq C\|f\|\|g\|,$$

where $\|\cdot\| = \|\cdot\|_{A^2}$, then by a general theorem from operator theory (based on the Riesz representation theorem for a bounded linear functional on a Hilbert space), there exists a bounded operator $T(\mu)$ on A^2 such that

$$(T(\mu)f,g) = t(f,g)$$

for all analytic polynomials f, g. When μ is absolutely continuous, $d\mu(z) = a(z)dA(z)$, we recover the old definition of the Toeplitz operator $T(a)$.

4.2. **Uniqueness of the symbol.**

Theorem 4.1. *Let μ_1, μ_2 be two signed measures in $\mathbb{D}$ such that $T(\mu_1)$, $T(\mu_2)$ are bounded. If $T(\mu_1) = T(\mu_2)$, then $\mu_1 = \mu_2$ a.e. on $\mathbb{D}$.*

Proof. It suffices to prove that $T(\mu) = 0$ implies $\mu = 0$. We will prove that

$$\int_{\mathbb{D}} F(z)d\mu(z) = 0$$

for any function F continuous on $\mathrm{clos}(\mathbb{D})$; this will imply $\mu = 0$. It suffices to check the last relation for all polynomials $F = F(x,y)$, $z = x+iy$. Consider a monomial:

$$\int_{\mathbb{D}} x^n y^m d\mu(z) = \int_{\mathbb{D}} \left(\frac{z+\overline{z}}{2}\right)^n \left(\frac{z-\overline{z}}{2i}\right)^m d\mu(z). \tag{4.1}$$

Expanding the r.h.s., we can rewrite this as a linear combination of terms of the type

$$\int_{\mathbb{D}} z^k \overline{z}^\ell d\mu(z) = (T(\mu)z^k, z^\ell) = 0$$

by assumption. Thus, (4.1) vanishes for all n, m, and we obtain the required statement. □

In fact, a much stronger statement is true:

Proposition 4.2. *Let μ be a signed measure on $\mathbb{D}$ such that $T(\mu)$ is a finite rank operator. Then μ is a finite linear combination of point masses on $\mathbb{D}$.*

The proof of this is very non-trivial; see the paper *Finite rank Toeplitz operators on the Bergman space* by D.Luecking in Proc. Amer. Math. Soc. 136 (2008), no. 5, 1717–1723.

4.3. **Boundedness: example.** In order to show that the trivial norm bound $\|T(a)\| \leq \|a\|_\infty$ is far from being sharp, let us consider an example.

Let ν be a finite measure on the interval $[0, 1]$ such that $\operatorname{supp} \nu \subset [0, a]$ with some $a < 1$. Set

$$d\mu(z) = d\nu(r)d\theta, \quad z = re^{i\theta};$$

i.e. μ is a rotationally symmetric measure. It is straightforward to see that $(T(\mu)z^n, z^m) = 0$ for $n \neq m$, and so the operator $T(\mu)$ is diagonal in the standard basis $\{(n + 1)^{1/2} z^n\}_{n \geq 0}$ in A^2. It follows that $T(\mu)$ is unitarily equivalent to the operator of multiplication by the sequence

$$r_n = (n + 1)(T(\mu)z^n, z^n) = 2\pi(n + 1) \int_0^a r^{2n} d\nu(r).$$

By our assumption on the support of ν, we have

$$r_n \leq 2\pi(n + 1)a^n \nu([0, a]) \to 0,$$

as $n \to \infty$, and so the operator $T(\mu)$ is compact.

This example shows that the symbol may be very far from a bounded function, yet $T(\mu)$ will be bounded. The crucial condition that ensures boundedness in this example is that the symbol vanishes near the boundary of the disk.

This example also shows that, in contrast to the Hardy-Toeplitz case, there are many non-trivial compact Bergman-Toeplitz operators.

4.4. **Berezin transform.** Recall that the reproducing kernel on Bergman space A^2 is given by $k_w(z) = (1 - z\overline{w})^{-2}$ (see Section 1.4). By definition,

$$f(w) = (f, k_w), \quad w \in \mathbb{D}$$

for any $f \in A^2$. We will need the formula for the norm of k_w:

$$\|k_w\|^2 = (k_w, k_w) = k_w(w) = (1 - |w|^2)^{-2}, \quad w \in \mathbb{D}.$$

Definition. Let μ be a finite measure on $\mathbb{D}$. *The Berezin transform of* μ is defined by

$$B\mu(w) = \frac{(T(\mu)k_w, k_w)}{\|k_w\|^2} = \int_{\mathbb{D}} \frac{(1 - |w|^2)^2}{|1 - z\overline{w}|^4} d\mu(z), \quad w \in \mathbb{D}.$$

Similarly, for $a \in L^1(\mathbb{D})$, the Berezin transform of a is defined by

$$Ba(w) = \frac{(T(a)k_w, k_w)}{\|k_w\|^2} = \int_{\mathbb{D}} \frac{(1 - |w|^2)^2}{|1 - z\overline{w}|^4} a(z)dA(z), \quad w \in \mathbb{D}.$$

The Berezin transform contains a lot of information about the properties of the corresponding Toeplitz operator. However, extracting this information turns out to be a very subtle problem.

We will need the following simple

Lemma 4.3. *Let h be a harmonic function in $L^2(\mathbb{D})$; then $Bh(w) = h(w)$ for all $w \in \mathbb{D}$.*

Proof. A function h in $L^2(\mathbb{D})$ is harmonic iff it can be represented as

$$h(z) = h_0 + \sum_{n\geq 1} h_n z^n + \sum_{n\geq 1} h_{-n}\overline{z}^n$$

with $\sum_{n\in\mathbb{Z}}(n+1)|h_n|^2 < \infty$. Clearly, we can write $h = u + \overline{v}$ with $u, v \in A^2$. We have

$$(uk_w, k_w) = u(w)k_w(w) = u(w)(1 - |w|^2)^{-2}$$

and similarly

$$(\overline{v}k_w, k_w) = \overline{(vk_w, k_w)} = \overline{v(w)}(1 - |w|^2)^{-2}.$$

Combining these identities, we obtain

$$(hk_w, k_w) = h(z)(1 - |w|^2)^{-2},$$

which is equivalent to $Bh(w) = h(w)$. □

4.5. **The norm.** As discussed above, the trivial norm estimate

$$\|T(a)\| \leq \|a\|_\infty$$

in general is not sharp. Some estimates for the norm are known, but it seems that the precise expression for the norm of a Bergman-Toeplitz operator is not available even for continuous symbols. We will treat only a very special case when the above trivial estimate turns out to be sharp.

Theorem 4.4. *Let a be harmonic in $\mathbb{D}$ and continuous in* clos$(\mathbb{D})$. *Then*

$$\|T(a)\| = \|a\|_\infty.$$

Proof. For any $w \in \mathbb{D}$ we have

$$|(T(a)k_w, k_w)| \leq \|T(a)\|\|k_w\|^2,$$

which can be rewritten as the estimate for the Berezin transform of a:

$$|Ba(w)| \leq \|T(a)\|, \quad w \in \mathbb{D}.$$

By Lemma 4.3, we have $Ba = a$ and therefore we obtain

$$\|a\|_\infty = \max_{|w|\leq 1}|a(w)| \leq \|T(a)\|,$$

which proves the claim. □

Lecture 4:

Of course, the same argument shows the bound

$$\sup_{\mathbb{D}}|Ba(w)| \leq \|T(a)\|$$

for *all* symbols a, but it is not easy to extract information about a from this bound, because the Berezin transform involves some averaging.

4.6. **Compactness.** First we give a generalization of the example in Section 4.3.

Theorem 4.5. *Let μ be a finite measure on $\mathbb{D}$ supported in the disk $\mathbb{D}_a = \{z : |z| < a\}$ with some $a < 1$. Then $T(\mu)$ is compact.*

We need a classical fact about integral operators with continuous kernels, which we state for $L^2(\mathbb{D})$.

Lemma 4.6. *Let K be an integral operator in $L^2(\mathbb{D})$ with an integral kernel $K = K(z, z')$ that is continuous in z, z' inside the closed unit disc* clos$(\mathbb{D})$*. Then K is compact and*

$$\|K\| \leq \sup_{\mathbb{D}\times\mathbb{D}} |K(z, z')|.$$

Proof. The norm bound is straightforward:

$$\left| \int_{\mathbb{D}} \int_{\mathbb{D}} K(z, z') f(z) \overline{g(z')} dA(z) dA(z') \right| \\ \leq \sup_{\mathbb{D}\times\mathbb{D}} |K(z, z')| \int_{\mathbb{D}} |f(z)| dA(z) \int_{\mathbb{D}} |g(z')| dA(z') \leq \sup_{\mathbb{D}\times\mathbb{D}} |K(z, z')| \|f\| \|g\|.$$

If $K(z, z')$ is a polynomial, then K is a finite rank operator. By the Weierstrass approximation theorem, we can approximate the kernel $K(z, z')$ by polynomials uniformly in $\mathbb{D} \times \mathbb{D}$. This yields an approximation of K in the operator norm by finite rank operators. Thus, K is compact. □

Proof of Theorem 4.5. Consider the quadratic form of $T(\mu)$:

$$\begin{aligned} (T(\mu) f, g) &= \int_{\mathbb{D}} f(z) \overline{g(z)} d\mu(z) \\ &= \int_{\mathbb{D}} (f, k_z) \overline{(g, k_z)} d\mu(z) = \int_{\mathbb{D}} \int_{\mathbb{D}} K(w, w') f(w) \overline{g(w')} dA(w) dA(w'), \end{aligned}$$

where

$$K(w, w') = \int_{\mathbb{D}} \overline{k_z(w)} k_z(w') d\mu(z).$$

Let K be the integral operator in $L^2(\mathbb{D})$ with the integral kernel $K(w, w')$. Elementary analysis shows that $K(w, w')$ is continuous in w, w', and so by Lemma the operator K is compact. We have

$$T(\mu) = \Pi_+ K \Pi_+^*,$$

where $\Pi_+ : L^2(\mathbb{D}) \to A^2$ is the Bergman projection. Thus, $T(\mu)$ is compact. □

The vanishing of the symbol at the boundary of the unit disk is key to compactness. In order to demonstrate this, let us consider a simple class of symbols.

Theorem 4.7. *Let a be a continuous function on the closed unit disk. Then $T(a)$ is compact iff $a|_{\mathbb{T}} = 0$.*

We need a simple

Lemma 4.8. *Let a be a continuous function on the closed unit disk. Then for any $z \in \mathbb{T}$, we have*

$$a(z) = \lim_{r\to 1-0} Ba(rz).$$

The proof is outlined in the exercises.

Proof of Theorem 4.7. (1) Suppose that $a|_{\mathbb{T}} = 0$. Then we can find a sequence of functions a_n, vanishing for $|z| \geq 1 - \frac{1}{n}$ and such that $\|a - a_n\|_\infty \to 0$ as $n \to \infty$. For each n, the operator $T(a_n)$ is compact by Theorem 4.5. We also have the norm convergence

$$\|T(a) - T(a_n)\|_\infty \leq \|a - a_n\|_\infty \to 0$$

as $n \to \infty$. Thus, $T(a)$ is compact.

(2) Suppose that $T(a)$ is compact. Consider the family of elements $(1-|w|^2)k_w \in A^2$, parameterised by $w \in \mathbb{D}$. These elements have norm one in A^2. As $|w| \to 1$, this family converges weakly to zero. Indeed, for any polynomial f, we have

$$(1 - |w|^2)(f, k_w) = (1 - |w|^2)f(w) \to 0$$

as $|w| \to 1$. Since $T(a)$ is compact, it follows that

$$(1 - |w|^2)^2(T(a)k_w, k_w) = Ba(w) \to 0$$

as $|w| \to 1$. Now the result follows from the previous Lemma. □

4.7. **Exercises.**

Exercise 4.1. Prove Lemma 4.8. Proceed as follows. For $z \in \mathbb{T}$, consider

$$Ba(rz) = \int_{\mathbb{D}} K_r(z, w)a(w)dA(w), \quad K_r(z, w) = \frac{(1 - r^2)^2}{|1 - rw\overline{z}|^4}.$$

Check that the kernel K_r satisfies the following properties:
(i) $K_r(z, w) \geq 0$;
(ii) $\int_{\mathbb{D}} K_r(z, w)dA(w) = 1$;
(iii) $\sup\{K_r(z, w) : |w - z| > \delta\} \to 0$ as $r \to 0$ for any $\delta > 0$.
Now proceed as in Exercise 1.2.

Exercise 4.2. Let a be continuous in the closed unit disk. Prove that $a(\mathbb{T}) \subset \sigma_{\mathrm{ess}}(T(a))$ by constructing a Weyl sequence for every point $\lambda \in a(\mathbb{T})$.

5. Hankel operators on Hardy space

5.1. **Non-uniqueness of the symbol.** Recall that the Hankel operator $H(a)$ on the Hardy space H^2_+ is defined in Section 1.8. The "matrix" of $H(a)$ in the standard basis $\{z^n\}_{n=0}^\infty$ is $\{\hat{a}(n + m)\}_{n,m=0}^\infty$. Since this involves only the Fourier coefficients $\hat{a}(k)$ with $k \geq 0$, we see that

$$H(a) = 0 \Leftrightarrow a \in H^\infty_-.$$

In particular, the symbol of a Hankel operator is not unique, but is defined up to an additive term from H^∞_-.

Remark. Some authors require that the symbol a is analytic, i.e. $\hat{a}(k) = 0$ for $k < 0$; of course, this condition uniquely specifies the symbol. We will not follow this approach.

5.2. **The norm.** As already discussed, we have the estimate

$$\|H(a)\| \leq \|a\|_\infty.$$

From the previous discussion it follows that

$$\|H(a)\| \leq \|a - \varphi\|_\infty \quad \forall \varphi \in H^\infty_-,$$

which implies that

$$\|H(a)\| \leq \mathrm{dist}_{L^\infty}(a, H^\infty_-).$$

It turns out that this inequality is actually an identity.

Theorem 5.1 (Z.Nehari 1957)**.** *Let $a \in L^\infty(\mathbb{T})$. Then there exists a function $b \in L^\infty(\mathbb{T})$ such that $b - a \in H^\infty_-$ and*

$$\|H(a)\| = \|b\|_\infty.$$

Corollary 5.2. *If $a \in L^\infty$, then*

$$\|H(a)\| = \mathrm{dist}_{L^\infty}(a, H^\infty_-). \tag{5.1}$$

Proof. We only need to prove the "$\geq$" inequality. Let b be as in the Theorem; then $a = b + \varphi$ with $\varphi \in H^\infty_-$, so

$$\mathrm{dist}_{L^\infty}(a, H^\infty_-) \leq \|a - \varphi\|_\infty = \|b\|_\infty = \|H(a)\|.$$

□

In order to prove Nehari's theorem, we will use the following fact from the theory of Hardy spaces:

Proposition 5.3. *Let $h \in H^1(\mathbb{T})$; then there exist $f, g \in H^2(\mathbb{T})$ such that $\|h\|_1 = \|f\|_2 \|g\|_2$.*

Proof of Nehari's theorem. Consider the anti-linear functional $\mathcal{L}$, defined on H^1 by

$$\mathcal{L}(h) = \int_{-\pi}^{\pi} a(e^{i\theta}) \overline{h(e^{i\theta})} \frac{d\theta}{2\pi}.$$

Fix $h \in H^1$; by Proposition above, we can write $h = fg$ with $f, g \in H^2$. Denote $f_1(e^{i\theta}) = \overline{f(e^{-i\theta})}$; then $f_1 \in H^2$ and $\|f_1\|_2 = \|f\|_2$. We have

$$\overline{h(e^{i\theta})} = \overline{g(e^{i\theta}) f(e^{i\theta})} = \overline{g(e^{i\theta})} f_1(e^{-i\theta}),$$

and so

$$\mathcal{L}(h) = \int_{-\pi}^{\pi} a(e^{i\theta}) f_1(e^{-i\theta}) \overline{g(e^{i\theta})} \frac{d\theta}{2\pi} = (H(a) f_1, g).$$

It follows that

$$|\mathcal{L}(h)| \leq \|H(a)\| \|f_1\|_2 \|g\|_2 = \|H(a)\| \|h\|_1,$$

and so $\mathcal{L}$ is a bounded linear functional on H^1 with the norm $\leq \|H(a)\|$. Since $H^1 \subset L^1$, by the Hahn-Banach theorem, $\mathcal{L}$ can be extended to a bounded linear functional on L^1 with the same norm. As $(L^1)^* \simeq L^\infty$, we obtain that there exists $b \in L^\infty$ with $\|b\|_\infty \leq \|H(a)\|$ such that

$$\mathcal{L}(h) = \int_{-\pi}^{\pi} b(e^{i\theta}) \overline{h(e^{i\theta})} \frac{d\theta}{2\pi}.$$

Repeating the above argument, we see that $H(a) = H(b)$, and therefore $b - a \in H^\infty_-$. Thus, we also have $\|H(a)\| \leq \|b\|_\infty$. □

In fact, Nehari's theorem can be stated without the *a priori* assumption that $a \in L^\infty$; all that is required is the boundedness of the *operator* $H(a)$. This can be done more cleanly in terms of the "matrix representation" for $H(a)$.

Let $\{\alpha_j\}_{j=0}^\infty$ be a sequence of complex numbers. We denote by $\Gamma(\alpha)$ the operator in $\ell^2(\mathbb{Z}_+)$, given by the "infinite matrix" $\{\alpha_{j+k}\}_{j,k\geq 0}$. More precisely, we start with a sesquilinear form

$$\Gamma(x, y) = \sum_{j,k\geq 0} \alpha_{j+k} x_j \overline{y_k}$$

on the set of all finite sequences x, y. If the form satisfies the bound

$$|\Gamma(x, y)| \leq A \|x\|_{\ell^2} \|y\|_{\ell^2},$$

then there exists a bounded operator $\Gamma(\alpha)$ in ℓ^2 such that

$$\Gamma(x, y) = (\Gamma(\alpha)x, y)$$

for all finite sequences x, y, and the smallest possible constant A in the estimate above coincides with the operator norm of $\Gamma(\alpha)$.

Theorem 5.4 (Nehari, version 2). *A Hankel matrix $\Gamma(\alpha)$ in $\ell^2(\mathbb{Z}_+)$ is bounded iff there exists a function $a \in L^\infty(\mathbb{T})$ such that*

$$\hat{a}(k) = \alpha_k, \quad k \geq 0.$$

In this case

$$\|\Gamma(\alpha)\| = \inf\{\|a\|_\infty : \hat{a}(k) = \alpha_k, \quad k \geq 0\}$$

For the proof, see, e.g. the book *Hankel operators and their applications* by V.Peller.

Example 5.5. Consider the sequence $\alpha_n = 1/(n+1)$. The corresponding operator $\Gamma(\alpha)$ is called *the Hilbert matrix*. The symbol

$$b(z) = \sum_{n=0}^{\infty} \frac{1}{n+1} z^n$$

is not bounded on $\mathbb{T}$, as can be seen by considering $z \to 1$. However, one can choose a bounded symbol for the Hilbert matrix. Take

$$a(e^{i\theta}) = ie^{-i\theta}(\pi - \theta), \quad \theta \in [0, 2\pi).$$

Then (exercise)

$$\hat{a}(n) = 1/(n+1), \quad n \geq 0.$$

It follows that the Hilbert matrix is bounded and

$$\|\Gamma(\alpha)\| \leq \pi.$$

In fact, the norm of the Hilbert matrix exactly equals to π. It is known that the Hilbert matrix is not compact, and its spectrum coincides with the interval $[0, \pi]$.

Remark 5.6. The previous example illustrates the following point: *The operator P_+ is not bounded in L^∞* (although it is bounded in L^p for all $1 < p < \infty$). Indeed, for a function $a \in L^\infty$ as in the example, the function $b = P_+ a$ is unbounded.

5.3. **Finite rank Hankel operators.** One of the oldest theorems in the theory of Hankel operators is

Theorem 5.7 (L.Kronecker 1881)**.** *Let $a \in L^\infty$; then the Hankel operator $H(a)$ has a finite rank if and only if $P_+ a$ is a rational function.*

Remark.

(1) A rational function is a function of the form p/q, where p and q are polynomials in z.
(2) Kronecker's theorem also says that the rank of $H(a)$ equals to $\max\{\deg p, \deg q\}$, although we will not prove this.
(3) Since $P_+ a = p/q$ is analytic in $\mathbb{D}$, it can have poles only outside the closed unit disk.
(4) As in Nehari's theorem, it is not necessary to assume a priori that $a \in L^\infty$. Another way of stating Kronecker's theorem is to say that if the operator $\Gamma(\alpha) = \{\alpha_{n+m}\}_{n,m\geq 0}$ is bounded in ℓ^2 and has a finite rank, then the function
$$\sum_{n\geq 0} \alpha_n z^n$$
is rational.

Lecture 5:

Proof of Kronecker's theorem. 1. Let us prove that if $a = p/q$ is a rational function with poles outside the closed unit disk, then $H(a)$ is a bounded finite rank operator. Using a partial fraction decomposition, we can represent a as a finite sum

$$a(z) = a_0(z) + \sum_{k,m} \frac{c_{k,m}}{(z - z_k)^m},$$

where a_0 is a polynomial, $|z_k| > 1$, $m \in \mathbb{N}$, and $c_{k,m}$ are complex numbers. It is clear that $H(a_0)$ is a finite rank operator. Thus, it suffices to prove that for each symbol of the form

$$b_\zeta(z) = (z-\zeta)^{-m}, \quad m \in \mathbb{N}, \quad |\zeta| > 1,$$

the operator $H(b_\zeta)$ has a finite rank.

Consider first the case $m = 1$. Then

$$b_\zeta(z) = (z-\zeta)^{-1} = -\zeta^{-1}(1 - z\zeta^{-1})^{-1} = -\sum_{n=0}^{\infty} \zeta^{-n-1} z^n,$$

and so the matrix elements of the Hankel operator in $\ell^2(\mathbb{Z}_+)$ corresponding to $H(b_\zeta)$ are

$$(H(b_\zeta)z^n, z^k) = -\zeta^{-1}\zeta^{-n}\zeta^{-k}.$$

Denoting this matrix by Γ_ζ, we see that it can be represented as a rank one operator in $\ell^2(\mathbb{Z}_+)$, viz.

$$\Gamma_\zeta = -\zeta^{-1}(\cdot, \overline{\zeta}^{-n})\zeta^{-n}. \tag{5.2}$$

Next, we have

$$H((z-\zeta)^{-m-1}) = \frac{1}{m!}\left(\frac{d}{d\zeta}\right)^m H((z-\zeta)^{-1}).$$

It follows that the matrix of $H((z-\zeta)^{-m-1})$ is given by

$$\frac{1}{m!}\left(\frac{d}{d\zeta}\right)^m \Gamma_\zeta.$$

Differentiating (5.2), we see that this is a finite rank operator.

2. Let $\{\alpha_j\}_{j\geq 0}$ be a sequence of complex numbers and let $\Gamma(\alpha)$ be the corresponding Hankel operator in $\ell^2(\mathbb{Z}_+)$. We denote by $a(z)$ the formal power series

$$a(z) = \sum_{j\geq 0} \alpha_j z^j.$$

Assume that $\Gamma(\alpha)$ is bounded and has a finite rank. The boundedness of $\Gamma(\alpha)$ immediately implies that the sequence $\{\alpha_j\}$ is bounded, and therefore the series in the definition of $a(z)$ converges for $|z| < 1$. If $\operatorname{rank}\Gamma(\alpha) = n$, then the first n rows of the matrix $\Gamma(\alpha)$ are linearly dependent. This can be written as follows:

$$c_0 a(z) + c_1 P_+(\overline{z}a(z)) + c_2 P_+(\overline{z}^2 a(z)) + \cdots + c_n P_+(\overline{z}^n a(z)) = 0, \quad z \in \mathbb{T}, \tag{5.3}$$

where $c_0, \dots, c_n$ are some complex numbers not simultaneously equal zero. Notice that for any k, we have

$$z^k P_+(\overline{z}^k a(z)) = a(z) + p_k(z),$$

where $p_k(z)$ is a polynomial of degree $\leq k$. Now let us multiply (5.3) by z^n:

$$\sum_{k=0}^{n} c_k z^n P_+(\overline{z}^k a(z)) = 0.$$

This can be rewritten as

$$\sum_{k=0}^{n} c_k z^{n-k} a(z) = p(z),$$

where p is a polynomial of degree $\leq n$. Thus, denoting

$$q(z) = \sum_{k=0}^{n} c_k z^{n-k},$$

we obtain $a(z) = p(z)/q(z)$. Finally, since $a(z)$ is analytic in $\mathbb{D}$, the rational function p/q may not have poles in $\mathbb{D}$. A separate argument (see Exercises) shows that p/q may not have poles on the unit circle $\mathbb{T}$ either. Thus, $a = p/q$ is a bounded analytic function in $\mathbb{D}$, and in particular $P_+ a = a$. □

5.4. **Compactness.** We start with a simple statement

Theorem 5.8. *Let $a \in C(\mathbb{T})$; then $H(a)$ is compact.*

Essentially, this is the same statement as Lemma 3.11.

Proof. There exists a sequence of polynomials a_n such that $\|a - a_n\|_\infty \to 0$ as $n \to \infty$. For each polynomial a_n, the operator $H(a_n)$ has a finite rank. Thus, $H(a)$ can be approximated by finite rank operators in the operator norm, hence $H(a)$ is compact. □

A deep and non-trivial fact is the converse statement:

Theorem 5.9 (P.Hartman, 1958)**.** *Let $a \in L^\infty$; if $H(a)$ is compact, then there exists $b \in C(\mathbb{T})$ such that $H(a) = H(b)$.*

We need two lemmas: a function theoretic one and an operator theoretic one.

Lemma 5.10. *Let*

$$C + H^\infty_- = \{p + g : p \in C(\mathbb{T}), g \in H^\infty_-(\mathbb{T})\}.$$

Then $C + H^\infty_-$ is closed in $L^\infty(\mathbb{T})$.

This lemma is a little technical and we postpone its proof until the end of this section.

Lemma 5.11. *Let S_n be a sequence of bounded operators such that $S_n \to 0$ strongly as $n \to \infty$. Then for any compact operator K, $\|S_n K\| \to 0$ as $n \to \infty$.*

The proof is outlined in the exercises.

Proof of Hartman's theorem. Let S be the operator of multiplication by z in H^2_+, and let S^* be its adjoint. It is easy to see that $(S^*)^n \to 0$ strongly in H^2. It follows that

$$\|H(a)S^n\| = \|(S^*)^n H(a)^*\| \to 0, \quad n \to \infty \tag{5.4}$$

by the lemma. Next, we have

$$H(a)S^n f = P_+(aJ(z^n f)) = P_+(a\overline{z}^n Jf)$$

and so $H(a)S^n = H(a\overline{z}^n)$. Using (5.1), it follows that

$$\|H(a)S^n\| = \|H(\overline{z}^n a)\| = \operatorname{dist}_{L^\infty}(\overline{z}^n a, H^\infty_-) = \operatorname{dist}_{L^\infty}(a, z^n H^\infty_-).$$

Combining this with (5.4), we get

$$\operatorname{dist}_{L^\infty}(a, z^n H^\infty_-) \to 0, \quad n \to \infty.$$

It is easy to see that $z^n H^\infty_- \subset C + H^\infty_-$, and so

$$\operatorname{dist}_{L^\infty}(a, C + H^\infty_-) \leq \operatorname{dist}_{L^\infty}(a, z^n H^\infty_-);$$

so it follows that

$$\operatorname{dist}_{L^\infty}(a, C + H^\infty_-) = 0.$$

Since (by Lemma 5.10) the class $C + H^\infty_-$ is closed, we get that $a \in C + H^\infty_-$. Thus, a can be represented as $a = b + \varphi$ with $b \in C$ and $\varphi \in H^\infty_-$. Then $H(a) = H(b)$, as required. ◻

5.5. **Proof of Lemma 5.10.** First we prove an auxiliary statement. Denote for brevity $\overline{\mathbb{D}} = \operatorname{clos}(\mathbb{D})$.

Lemma 5.12. *Let* $f \in C(\mathbb{T})$. *Then*

$$\operatorname{dist}_{L^\infty}(f, H^\infty_-) = \operatorname{dist}_{L^\infty}(f, H^\infty_- \cap C(\overline{\mathbb{D}})).$$

Proof. Clearly, it suffices to prove that

$$\operatorname{dist}_{L^\infty}(f, H^\infty_- \cap C(\overline{\mathbb{D}})) \leq \operatorname{dist}_{L^\infty}(f, H^\infty_-).$$

Given $g \in L^\infty(\mathbb{T})$, we denote by $\widetilde{g}$ the harmonic extension of g into $\mathbb{D}$ and set $g^{(r)}(z) = \widetilde{g}(rz)$, $z \in \mathbb{T}$. Recall that (see Exercise 1.1)

$$\|g^{(r)}\|_\infty \leq \|g\|_\infty, \quad r < 1, \tag{5.5}$$

and

$$\|g^{(r)} - g\|_\infty \to 0, \quad r \to 1, \quad \forall g \in C(\mathbb{T}). \tag{5.6}$$

Let $f \in C(\mathbb{T})$. Denote $d = \operatorname{dist}(f, H^\infty_-)$; let us prove that

$$\operatorname{dist}_{L^\infty}(f, H^\infty_- \cap C(\overline{\mathbb{D}})) \leq d + \varepsilon \tag{5.7}$$

for any given $\varepsilon > 0$. There exists $h \in H^\infty_-$ such that $\|f - h\|_\infty \leq d + \varepsilon/2$. We have

$$\|f - h^{(r)}\|_\infty = \|f - f^{(r)} + f^{(r)} - h^{(r)}\|_\infty \leq \|f - f^{(r)}\|_\infty + \|f^{(r)} - h^{(r)}\|_\infty.$$

As $r \to 1$, we have $\|f - f^{(r)}\|_\infty \to 0$ by (5.6). Thus, we can choose r sufficiently close to 1 so that $\|f - f^{(r)}\|_\infty < \varepsilon/2$. On the other hand, by (5.5) we have

$$\|f^{(r)} - h^{(r)}\|_\infty \leq \|f - h\|_\infty \leq d + \varepsilon/2.$$

Combining this, we obtain (5.7). ◻

Proof of Lemma 5.10. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of elements of $C + H_-^\infty$; suppose that $\|a - a_n\|_\infty \to 0$ with some $a \in L^\infty$. We need to prove that $a \in C + H_-^\infty$. We have $a_n = p_n + g_n$ with some $p_n \in C$, $g_n \in H_-^\infty$. By selecting a subsequence, we may assume that the convergence $a_n \to a$ is sufficiently fast so that

$$\|a_{n+1} - a_n\|_\infty \leq 2^{-n}.$$

This means that

$$\|(p_{n+1} - p_n) + (g_{n+1} - g_n)\|_\infty \leq 2^{-n}. \tag{5.8}$$

Let us prove that we can replace the sequence $g_n \in H_-^\infty$ here by a sequence of functions from $H_-^\infty \cap C(\overline{\mathbb{D}})$; we will have to pay by slightly increasing the constant in the right hand side. That is, let us prove that for some sequence $q_n \in H_-^\infty \cap C(\overline{\mathbb{D}})$, we have

$$\|(p_{n+1} - p_n) + (q_{n+1} - q_n)\|_\infty \leq 2^{-n+1}. \tag{5.9}$$

Indeed, from (5.8) we have

$$\operatorname{dist}_{L^\infty}(p_{n+1} - p_n, H_-^\infty) \leq 2^{-n}.$$

By Lemma 5.12, it follows that

$$\operatorname{dist}_{L^\infty}(p_{n+1} - p_n, H_-^\infty \cap C(\overline{\mathbb{D}})) \leq 2^{-n},$$

and therefore there exist $Q_n \in H_-^\infty \cap C(\overline{\mathbb{D}})$ such that

$$\|(p_{n+1} - p_n) - Q_n\|_\infty \leq 2^{-n+1}.$$

Write $q_n = \sum_{k=1}^{n-1} Q_k$; then $Q_n = q_{n+1} - q_n$ and so we obtain (5.9).

From (5.9) we see that the sequence $p_n + q_n$ converges in L^∞; denote its limit by b. Since $p_n, q_n \in C$, we have $b \in C$.

Consider the quotient space L^∞ / H_-^∞, and let $\pi : L^\infty \to L^\infty / H_-^\infty$ be the natural projection. Clearly, π is a continuous map. As $p_n + g_n \to a$, we get $\pi(p_n) \to \pi(a)$ in L^∞ / H_-^∞. Since $q_n \in H_-^\infty$ we have $\pi(q_n) = 0$ and so $p_n + q_n \to b$ implies that $\pi(p_n) \to \pi(b)$ in L^∞ / H_-^∞. Thus, $\pi(a) = \pi(b)$, i.e. $a = b + h$ with some $h \in H_-^\infty$, as required. □

5.6. Exercises.

Exercise 5.1. Prove Lemma 5.11; proceed as follows. First prove the statement for the case when K is a rank one operator. Next, repeat for the case when K is a finite rank operator. Finally, complete the proof using the facts that (i) any strongly convergent sequence of operators is uniformly bounded; and (ii) if K is compact, then for any $\varepsilon > 0$ it can be represented as $K = K_1 + K_2$, where K_1 is a finite rank operator and $\|K_2\| \leq \varepsilon$.

Exercise 5.2. Give an example showing that under the hypothesis of Lemma 5.11, the statement $\|KS_n\| \to 0$ may be false (the order is important here!).

Exercise 5.3. Prove that in the second part of the proof of Kronecker's theorem, the function $a(z)$ is in H^∞. Proceed as follows. (i) Prove that if $\Gamma(\alpha)$ is a bounded operator of rank one, then $\sum_{j\geq 0}|\alpha_j| < \infty$. (ii) Extend this to the case when $\Gamma(\alpha)$ is a bounded operator of finite rank. (iii) Conclude that $a(z)$ is bounded uniformly in $\mathbb{D}$.