Probabilistic Weyl laws for quantized tori

The Brian Davies 65th Birthday Conference

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But the proof is different, hopefully *simpler*...

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which is just a discretization of f.

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$$\mathcal{F}_N(k,\ell) = \frac{\exp(2\pi i k\ell/N)}{\sqrt{N}} \,.$$

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$$\operatorname{tr} g_N = N \int_{\mathbb{T}} g + \mathcal{O}(N^{-\infty}),$$

for $g \in C^{\infty}(\mathbb{T}^n)$.

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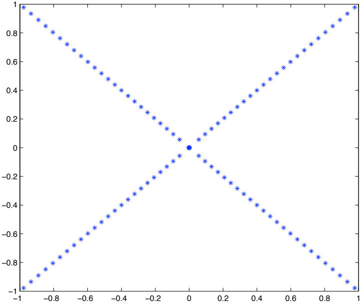
Spec (f_{100}):



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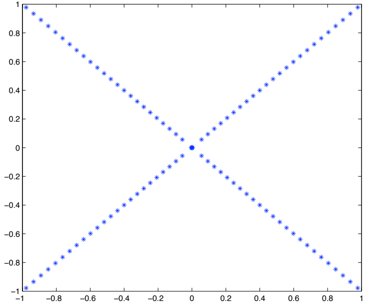

The spectrum is very unstable and its structure is related to the analytic continuation of f.

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The semiclassical parameter is still

$$h = \frac{1}{2\pi N},$$

in the sense that

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And it works the same for $\mathbb{T}^n$...

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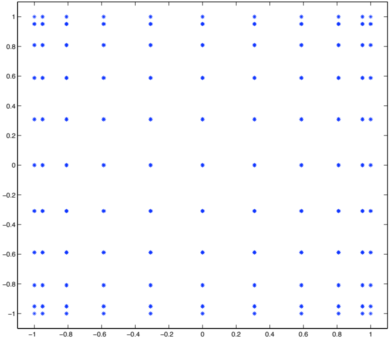

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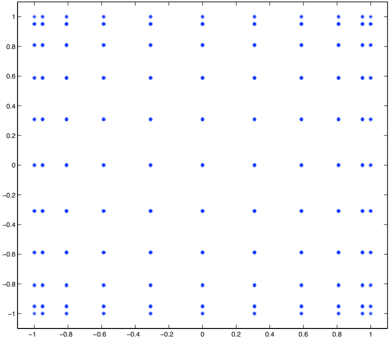

The spectrum is very stable and its structure is is related to F.

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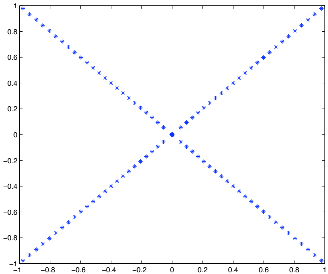

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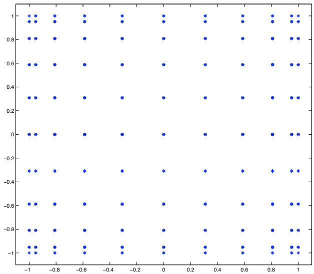
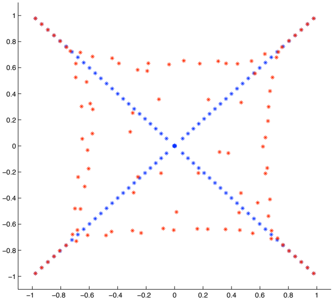
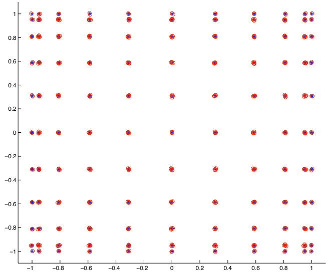


$\mathrm{Spec}\left(f_{100} + 100^{-2} R(100)\right)$



$\mathrm{Spec}\left(F_{20} + 20^{-2} R(400)\right)$



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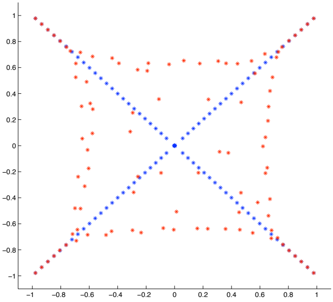

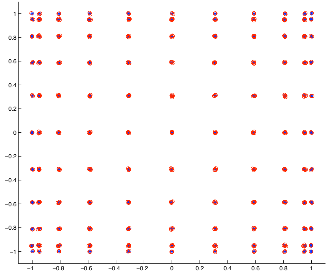

Instability

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$$\{\mathrm{Re}\, f, \mathrm{Im}\, f\}(x_0, \xi_0) < 0 \,.$$

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Suppose that $z_0 = f(x_0, \xi_0)$ and that

$$\{\mathrm{Re}\, f, \mathrm{Im}\, f\}(x_0, \xi_0) < 0 \,.$$

Then there exist $u_N \in \ell^2(\mathbb{Z}_N)$, $\|u_N\|_{\ell^2} = 1$, microlocalized to (x_0, ξ_0) such that

$$\|(f_N - z_0)u_N\| = \mathcal{O}(N^{-\infty}) \,.$$

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When f is real analytic $\mathcal{O}(N^{-\infty})$ can be replaced by $e^{-N/C}$.

In both cases, theorem states that z_0 is "almost" an eigenvalue.

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$$g \in C^\infty(\mathbb{T}), \quad g \equiv 0 \text{ near } (x_0, \xi_0) \implies \|g_N u_N\|_{\ell^2} = \mathcal{O}(N^{-\infty}).$$

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Here $u \in \mathcal{S}(\mathbb{R}^n)$ if $x^\beta \partial^\alpha u = \mathcal{O}(1)$.

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Key property

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This means that $\mathrm{Spec}\,(f_N + N^{-p} R_N(\omega))$, unlike $\mathrm{Spec}\,(f_N)$, displays a probabilistic Weyl law for the eigenvalues.

A numerical illustration:

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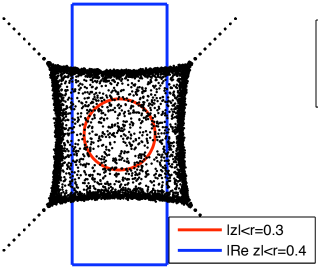

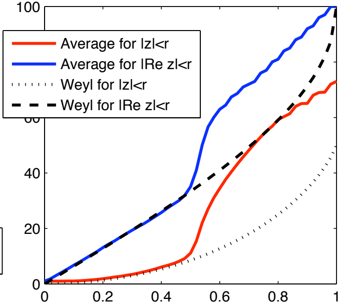

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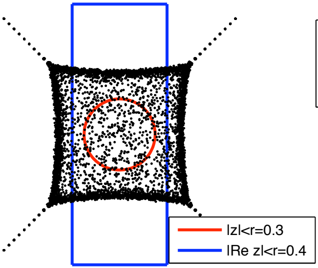

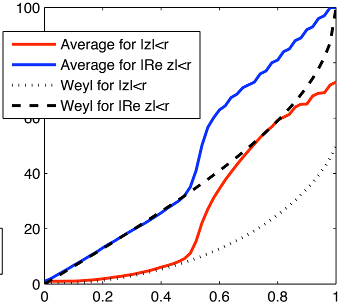

The left figure is from Embree-Trefethen.

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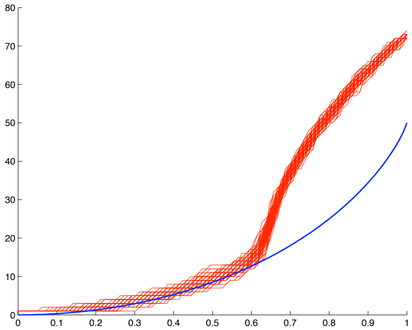

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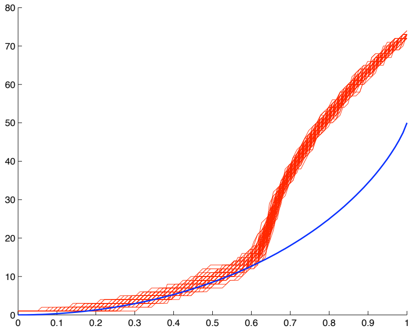

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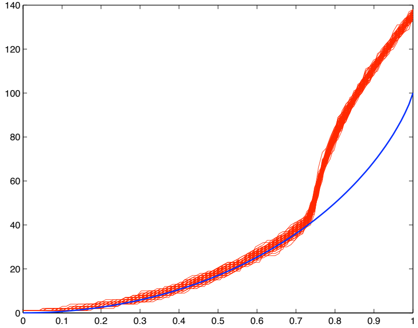

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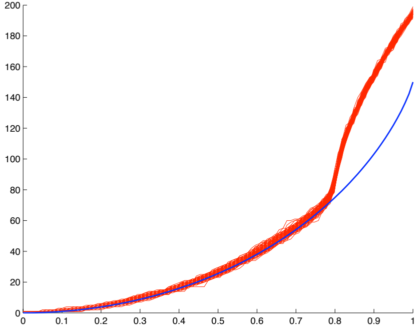

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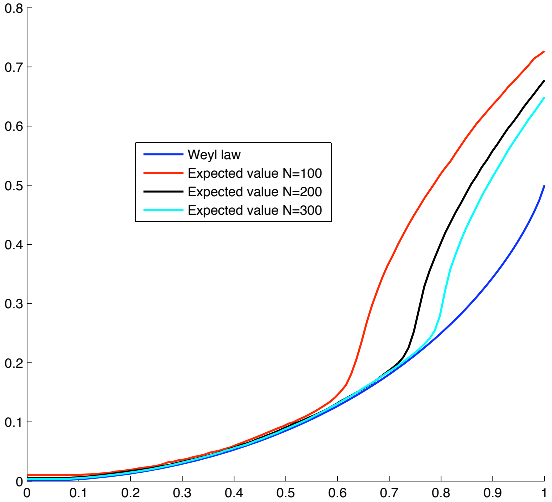

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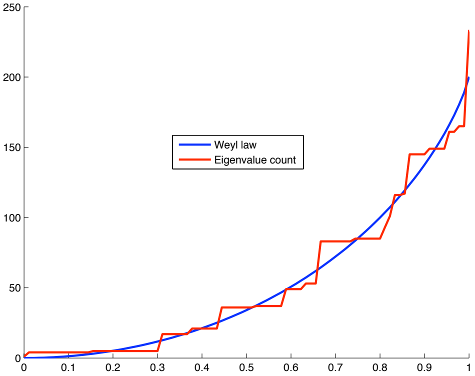

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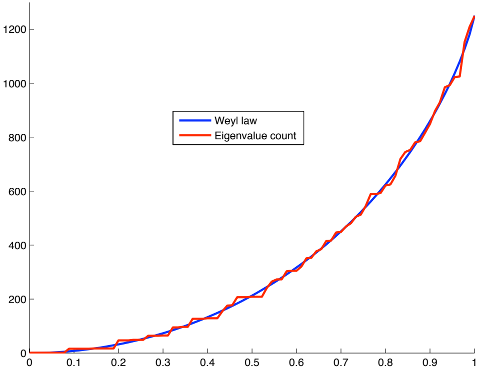

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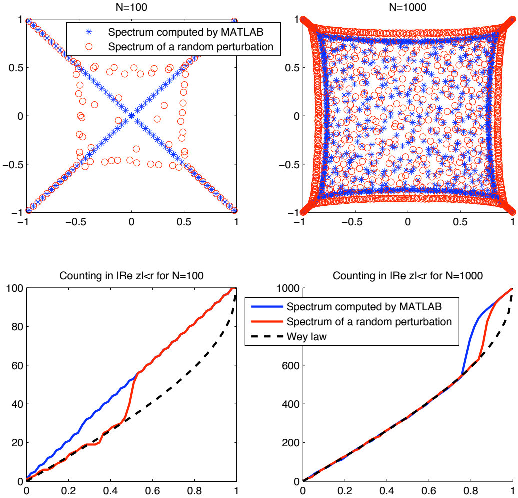

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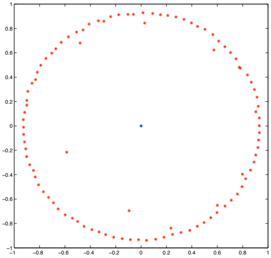

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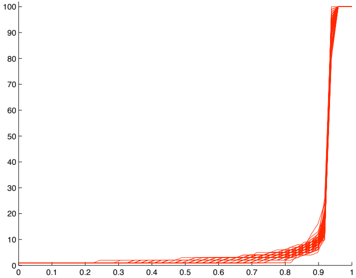

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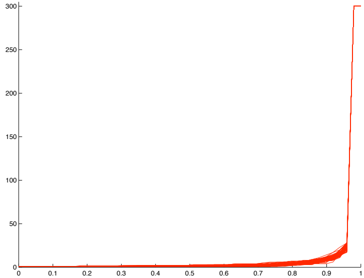

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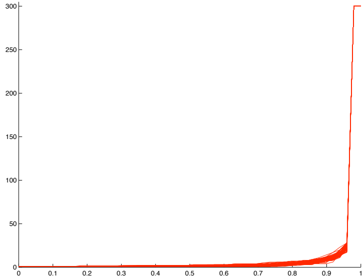

This is in agreement with the results of Davies-Hager and seems to hold for more general Toeplitz operators even though the theorem in the current form does not apply (Bordeaux Montrieux).

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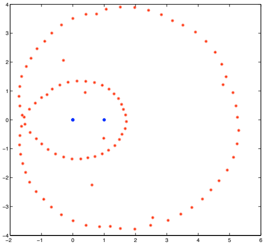

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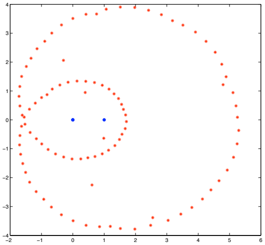

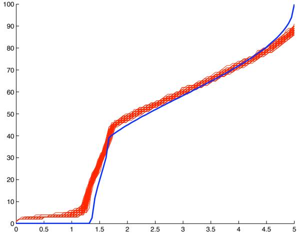

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- Condition (1) appears in Sjöstrand-Hager with $\kappa > 0$.
- If $df \wedge d\bar{f}\restriction_{f^{-1}(z)} \neq 0$ then it holds with $\kappa = 2$.
- For analytic functions function it always holds with some $\kappa > 0$: a version of a Łojasiewicz inequality (via resolutions of singularities by Bierstone-Milman and other analytic geometers).

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We can think of f as a map from $\mathbb{T}^n$ to $\mathbb{R}^2$ and the condition $df \wedge d\bar{f}\restriction_{f^{-1}(z)} \neq 0$ means that z is a *regular value* of f. Hence by the Morse-Sard Theorem, the set of z's at which $df \wedge d\bar{f}\restriction_{f^{-1}(z)} \neq 0$ holds has full Lebesgue measure in $\mathbb{C}$.

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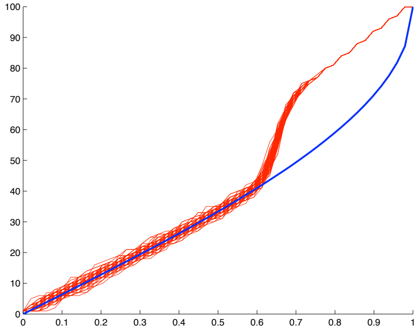

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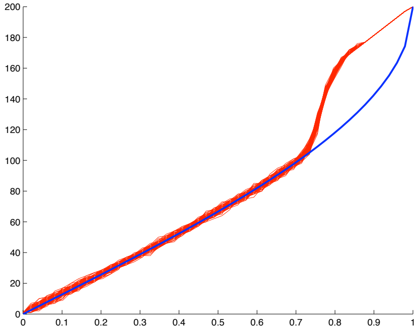

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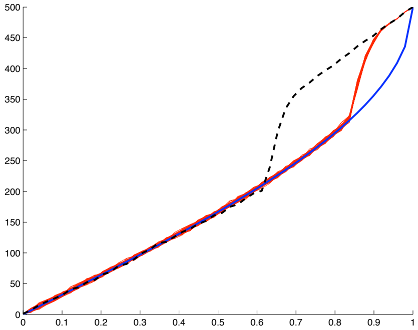

Here we added one more plot: numerically computed eigenvalues of f_{500}: the Weyl law appears for the numerically computed false eigenvalues!

"Proof of Theorem"

$$|\mathrm{Spec}\,(f_N)\cap\Omega| = \frac{1}{2\pi i}\int_{\partial\Omega}\mathrm{tr}(f_N - z)^{-1}dz$$

$$\text{``}{=}\text{''}\; N^n\frac{1}{2\pi i}\int_{\partial\Omega}\int_{\mathbb{T}^n}(f(\rho)-z)^{-1}d\mathcal{L}(\rho)dz + o(N^n)$$

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A random perturbation allows this argument to go through on the level of expected values.

We use the *singular value decomposition* of f_N to obtain a reduction to a nicer family of operators.

$$f_N = U_N S_N V_N^* \,,$$

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We note that

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This is obvious once we note that

$$\psi(f_N f_N^*/\alpha^2)U_N V_N^* = U_N \psi((S_N/\alpha)^2) V_N^*$$

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$$= \int_\gamma \mathrm{tr}(f_N + \alpha\psi(f_N f_N^*/\alpha^2) U_N V_N^* - z)^{-1} + \mathcal{O}\left(\frac{\delta}{\alpha} N^n + d \log\left(\frac{\alpha}{\delta}\right)\right),$$

where

$$d = \mathrm{rank}\,\psi\left(\frac{f_N f_N^*}{\alpha^2}\right).$$

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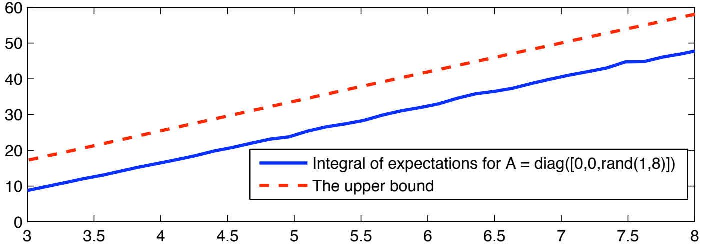

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which is a pseudodifferential operator in a slightly exotic class (similar to the one appearing in Hager-Sjöstrand 2008).

Using pseudodifferential calculus in that class we show that

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$$N^n \int_{\mathbb{T}^n} \frac{\bar{f}(\rho) d\mathcal{L}(\rho)}{|f(\rho)|^2 + \alpha^2\psi(|f(\rho)|^2/\alpha^2)} + \mathcal{O}(h^{-n+1-2\rho} + h^{-n+(\kappa-1)\rho}) =$$

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$$\beta = \frac{\kappa - 1}{\kappa + 1} .$$

Summing up over γ's covering $\partial\Omega$ and putting together all the error terms we get, for $p > p(n)$,

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